SI427 Problems and Solutions

Tut-1 :

1.(a) given $\mathcal{F}$ is a field, so ① $\phi \in \mathcal{F}$, $\Omega \in \mathcal{F}$
② $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
③ $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$

now if $n=2$, then $A_1 \cup A_2 \in \mathcal{F}$ (from ③)

lets suppose for $n=k$ it is true, ie
for $A_1, A_2, \dots A_k \in \mathcal{F}$
$A_1 \cup A_2 \cup \cdots A_k \in \mathcal{F}$

now for $n=k+1$ let $B = A_1 \cup A_2 \cup \cdots A_k$
$A_{k+1} = A_{k+1}$
then as for $n=k$ is true $B \in \mathcal{F}$
also $A_{k+1} \in \mathcal{F}$
as for $n=2$ true, $B \cup A_{k+1} \in \mathcal{F}$

If $n=k$ is true $\Rightarrow$ $n=k+1$ is also true

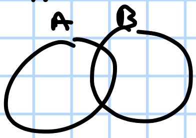
$\therefore$ By induction this is true.

Also as $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ ($\because$ ②)
$B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}$ ($\because$ ②)
$\Rightarrow$ $A^c \cup B^c \in \mathcal{F}$ ($\because$ ③)
$\Rightarrow$ $(A^c \cup B^c)^c \in \mathcal{F}$ ($\because$ ②)
$\Rightarrow$ $A \cap B \in \mathcal{F}$
so for $n=2$ $A_1 \in \mathcal{F}$ and $A_2 \in \mathcal{F}$
$\Rightarrow$ $A_1 \cap A_2 \in \mathcal{F}$

for $n=k$ lets suppose true:
$A_1, A_2, \dots A_k \in \mathcal{F}$
$\Rightarrow$ $A_1 \cap A_2 \cap \cdots A_k \in \mathcal{F}$

now for $n=k+1$: let
$B = A_1 \cap A_2 \dots A_k \in \mathcal{F}$ (as $n=k$ true)
$A_{n+1} = A_{n+1} \in \mathcal{F}$
as for $n=2$ true
$B \cap A_{n+1} \in \mathcal{F}$
$\therefore$ By induction true

(b) $A \setminus B = A \cap B^c$



now as $B \in \mathcal{F}$
$\Rightarrow B^c \in \mathcal{F}$
and from (a) if $A \in \mathcal{F}$ and $B^c \in \mathcal{F}$
$\Rightarrow A \cap B^c \in \mathcal{F}$
$\Rightarrow A \setminus B \in \mathcal{F}$

$A \Delta B = (A-B) \cup (B-A)$
as $A \setminus B \in \mathcal{F}$
similarly
$B \setminus A \in \mathcal{F}$
now as $A \setminus B \in \mathcal{F}$ and $B \setminus A \in \mathcal{F}$
$\Rightarrow (A \setminus B) \cup (B \setminus A) \in \mathcal{F}$
$\Rightarrow A \Delta B \in \mathcal{F}$

2. To prove: $\mathcal{F}_1, \mathcal{F}_2$ are two fields of subsets of Ω, then $\mathcal{F}_1 \cap \mathcal{F}_2$ is also a field.

proof: let $A \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $B \in \mathcal{F}_1 \cap \mathcal{F}_2$

$\Rightarrow A \in \mathcal{F}_1$ and $A \in \mathcal{F}_2$ $\quad \Rightarrow B \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$

now if $A \in \mathcal{F}_1 \Rightarrow A^c \in \mathcal{F}_1$
and $A \in \mathcal{F}_2 \Rightarrow A^c \in \mathcal{F}_2$
so $A^c \in \mathcal{F}_1$ and $\mathcal{F}_2$
$\Rightarrow A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$ — 1

as $A \cup B \in \mathcal{F}_1$, and $\mathcal{F}_2$
$\Rightarrow A \cup B \in \mathcal{F}_1 \cap \mathcal{F}_2$ — 2

as Ω and $\phi \in \mathcal{F}_1$ and $\mathcal{F}_2$
$\Omega \in \mathcal{F}_1 \cap \mathcal{F}_2$
$\phi \in \mathcal{F}_1 \cap \mathcal{F}_2$ — 3

as $\mathcal{F}_1 \cap \mathcal{F}_2$ satisfies 1, 2 and 3
$\mathcal{F}_1 \cap \mathcal{F}_2$ is a field.

3. $\mathcal{F}_1 = \{\Omega, \phi\}$ $\quad \mathcal{F}_2 = \{\Omega', \phi\}$ $\qquad \Omega = \{1, 2, \dots 6\}$, $\Omega' = \{H, T\}$

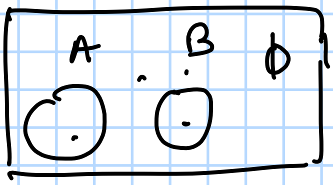
$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\Omega, \Omega', \phi\}$

as $\Omega \in \mathcal{F}_1$ and $\Omega' \in \mathcal{F}_1$
$\Omega \cup \Omega' \notin \mathcal{F}_1$

4. $\Omega = \{1, 2, \dots 4\}$ $\quad \{\{2\}, \{3\}\} = \mathcal{F}$

as $\Omega \in \mathcal{F}$ and ϕ
$\{\Omega, \{2\}, \{3\}, \phi\}$
also let $\{2\} = A$
$\{3\} = B$

$\{\Omega, \phi, A, B\}$



as $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
$B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}$
$A \cap B \in \mathcal{F} = \phi$
$A \cup B \in \mathcal{F}$
$(A \cup B)^c \in \mathcal{F}$

$\{\Omega, A, B, \phi, A^c, B^c, A \cup B, (A \cup B)^c\} \to \min \mathcal{F}$

Better method:

$C_1 = A \cap B^c$
$C_2 = A^c \cap B$
$C_3 = A \cap B$
$C_4 = (A \cup B)^c$

union of all is Ω.

generators or atoms of that field

see 16 then

5. $\Omega = [0, 1]$

$\{\phi, \overset{A}{[0, 1/2)}, \overset{B}{\{1\}}\}$

$\{\phi, [0, 1/2), \{1\}, \Omega\}$ $\longrightarrow$ $\{\phi, \Omega, A, B, A^c, B^c, A \cup B, (A \cup B)^c\}$

$A = [0, 1/2)$ $\quad A^c = [1/2, 1]$ $\quad A \cup B = [0, 1/2) \cup \{1\}$
$B = \{1\}$ $\quad B^c = [0, 1)$ $\quad A \cap B = \Omega$ $\quad (A \cap B)^c = \phi$
$(A \cup B)^c = [1/2, 1)$

6. $\mathcal{F} = \{A \subset \Omega \mid A \text{ is finite set}\}$

NO:

for $\Omega = \mathbb{N}$
$A = \{1\} \in \mathcal{F}$
but $A^c \notin \mathcal{F}$ as $A^c = \mathbb{N} - \{1\}$ Not finite.

7. $A_0 = \phi$

$A_n = \{1, 2, \ldots n\}$

$\mathcal{F} = \{A_n\}_{n=0}^{\infty} \cup \{\mathbb{N} \setminus A_n\}_{n=0}^{\infty} = \{\phi, \{1\}, \{1,2\}, \{1,2,3\}, \ldots\} \cup \{\mathbb{N}, \mathbb{N}-\{1\}, \ldots\}$

$= \{\phi, \{1\}, \{1,2\}, \{1,2,3\}, \ldots\} \cup \{\mathbb{N}, \{2,3,\ldots\}, \{3,4,\ldots\} \ldots\}$

$\mathcal{F}$ is not a field as for

$A = \{1,2,3\}$ $\quad A^c = \mathbb{N} - \{1,2,3\} \in \mathcal{F}$

$B = \mathbb{N} - \{1\}$ $\quad B^c = \{1\} \in \mathcal{F}$

$A^c \cup B^c = \{1, 4, 5 \ldots\} \notin \mathcal{F}$

$\therefore$ $\mathcal{F}$ is not a field

8. $\{A_i / i \in \mathbb{N}\}$ $\qquad (\cup_i A_i)^c = \cap_i A_i^c$

proof: let $\alpha \in \cap_i A_i^c$

$\Rightarrow \alpha \in A_1^c, \alpha \in A_2^c \ldots$

$\Rightarrow \alpha$ is not in $A_1, A_2, \ldots\ldots$

$\Rightarrow \alpha$ is not in any A_i $\forall i \in \mathbb{N}$

$\Rightarrow \alpha \in (\cup_i A_i)^c$

so $\quad \cap_i A_i^c \subseteq (\cup_i A_i)^c$ — ①

also if $\beta \in (\cup_i A_i)^c$

$\Rightarrow \beta$ is not in A_1 or A_2 or $A_3 \ldots$

$\Rightarrow \beta$ is in A_1^c and A_2^c and $A_3^c \ldots$

$\Rightarrow \beta \in \cap_i A_i^c$

$(\cup_i A_i)^c \subseteq \cap_i A_i^c$ — ②

from ① and ② $\quad (\cup_i A_i)^c = \cap_i A_i^c$

$(\cap_i A_i)^c = \cup_i A_i^c$

proof:

$\alpha \in (\cap_i A_i)^c$

$\Rightarrow \alpha$ is not in A_1 and $A_2 \ldots$

$\Rightarrow \alpha$ is in A_1^c or A_2^c or $A_3^c \ldots$

$\Rightarrow \alpha \in \cup_i A_i^c$

so $\quad (\cap_i A_i)^c \subseteq \cup_i A_i^c$ — ①

$\beta \in \cup_i A_i^c$

$\Rightarrow \beta$ is in not A_1 or not $A_2 \ldots$

$\Rightarrow \beta$ is not in A_1 and A_2 and $A_3 \ldots$

$\Rightarrow \beta \in (\cap_i A_i)^c$

$\cup_i A_i^c \subseteq (\cap_i A_i)^c$ — ②

from ① and ②

$\cup_i A_i^c = (\cap_i A_i)^c$

9. $\mathcal{F}_1$ $\mathcal{F}_2$ σ- fields of Ω $\qquad \mathcal{F}_1 \cap \mathcal{F}_2$ $\quad$ proof: let $A \in \mathcal{F}_1 \cap \mathcal{F}_2$ $\quad B \in \mathcal{F}_1 \cap \mathcal{F}_2$

then $A \in \mathcal{F}_1$ and $A \in \mathcal{F}_2$

$B \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$

as both σ-fields so
$\Omega \in \mathcal{F}_1$ and $\Omega \in \mathcal{F}_2$
$\Rightarrow \Omega \in \mathcal{F}_1 \cap \mathcal{F}_2$

also as $\phi \in \mathcal{F}_1$ and $\phi \in \mathcal{F}_2$
$\Rightarrow \phi \in \mathcal{F}_1 \cap \mathcal{F}_2$

also as $A \in \mathcal{F}_1 \cap \mathcal{F}_2$
$A \in \mathcal{F}_1$ and $A \in \mathcal{F}_2$
$\Rightarrow A^c \in \mathcal{F}_1$ and $A^c \in \mathcal{F}_2$
$\Rightarrow A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$

now if $A_1, A_2, \ldots \in \mathcal{F}_1 \cap \mathcal{F}_2$
then
all $\in \mathcal{F}_1$ and $\mathcal{F}_2$

as σ-fields
$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_2$
$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1 \cup \mathcal{F}_2$

10. $\mathcal{F}$ is finite then it is a sigma field as sigma fields: for $A_i \in \mathcal{F}$ $\forall i \in \mathbb{N}$
$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

but as $\mathcal{F}$ is finite union of all already belongs to $\mathcal{F}$. (Q1)

11. NO
$\mathcal{F} = \{A \subseteq \Omega \mid A \text{ or } A^c \text{ is finite}\}$ Name: finite-co-finite sets

for $\Omega = \mathbb{Z}$, now for
$\{1\} \in \mathcal{F}$
$\{2\} \in \mathcal{F}$
$\vdots$ but $\{1\} \cup \{2\} \cdots = \mathbb{N} \notin \mathcal{F}$ as $\mathbb{N}$ or $\mathbb{N}^c$ is not finite.

12. $\Omega = [0,1]$ $\mathcal{F}$ is σ-field on Ω
$\left[\frac{1}{n+1}, \frac{1}{n}\right] \in \mathcal{F}$ $\forall n \in \mathbb{N}$

to show: $\{0\} \in \mathcal{F}$; $\left(\frac{1}{n}, 1\right] \in \mathcal{F}$ $\forall n \in \mathbb{N}$
$\left\{\frac{1}{n} \mid n = 2, 3, \ldots\right\} \in \mathcal{F}$; $\left(0, \frac{1}{n}\right) \in \mathcal{F}$ $\forall n \in \mathbb{N}$

proof: as $\left[\frac{1}{n+1}, \frac{1}{n}\right] \in \mathcal{F}$
$\Rightarrow \left[\frac{1}{2}, 1\right] \in \mathcal{F}$, $\left[\frac{1}{3}, \frac{1}{2}\right] \in \mathcal{F}$, $\left[\frac{1}{4}, \frac{1}{3}\right] \in \mathcal{F}$ ....
union of all is:
$(0, 1] \in \mathcal{F}$ $\left(\because \forall \varepsilon \in (0,1), \exists n \in \mathbb{N} \text{ s.t } \varepsilon < \frac{1}{n}\right)$
then $(0,1]^c \in \mathcal{F} \Rightarrow \{0\} \in \mathcal{F}$

also as $\left[\frac{1}{n+1}, \frac{1}{n}\right] \in \mathcal{F}$ then $\left[\frac{1}{n+1}, \frac{1}{n}\right] \cup \left[\frac{1}{n+2}, \frac{1}{n+1}\right] \ldots$

$\Rightarrow (0, \frac{1}{n}] \in \mathcal{F}$ and as $\{0\} \in \mathcal{F}$

$\Rightarrow [0, \frac{1}{n}] \in \mathcal{F}$

now as $[0, 1] \in \mathcal{F}$ (even for $n=1$, as empty set)

$\Rightarrow (\frac{1}{n}, 1] \in \mathcal{F}$

also $[\frac{1}{n+1}, \frac{1}{n}] \in \mathcal{F}$

$\Rightarrow [\frac{1}{n}, \frac{1}{n-1}] \in \mathcal{F}$ for $n \in \{2, 3, \dots\}$

now $\Rightarrow [\frac{1}{n}, 1] \in \mathcal{F}$

as $[0, \frac{1}{n}] \in \mathcal{F}$ and $[\frac{1}{n}, 1] \in \mathcal{F}$

$\Rightarrow \{\frac{1}{n}\} \in \mathcal{F} \quad \forall n \in \{2, 3, \dots\}$

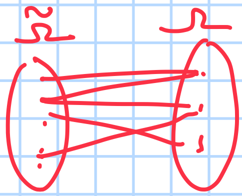
also as $(0, \frac{1}{n}] \in \mathcal{F}$ and $\{\frac{1}{n}\} \in \mathcal{F}$

$\forall n = 2, 3, \dots \quad (0, \frac{1}{n}) \in \mathcal{F}$

for $(0,1)$: as $[\frac{1}{n}, 1] \in \mathcal{F}$ for $n=1$
$\{1\} \in \mathcal{F}$

so $[0,1] - \{0\} - \{1\} \in \mathcal{F}$
$\Rightarrow (0,1) \in \mathcal{F}$

13. $\Omega, \tilde{\Omega}$ $\quad X : \tilde{\Omega} \to \Omega$
← Any function



bijective

$X^{-1}(\Omega) = \{\omega \in \tilde{\Omega} \mid X(\omega) = \Omega\} = \tilde{\Omega}$

if $\mathcal{F}$ is σ-field then
$\tilde{\mathcal{F}} = \{X^{-1}(A) \mid A \in \mathcal{F}\}$ is σ-field on $\tilde{\Omega}$.

if X is bijective then:
$\exists$ some $A \in \mathcal{F}$ s.t
$X^{-1}(A) = \Omega$
also $\exists$ some $B \in \mathcal{F}$ s.t
$X^{-1}(B) = \phi$

also if $A_i \in \mathcal{F}$, $\cup A_i \in \mathcal{F}$
$X^{-1}(A_i) \in \tilde{\mathcal{F}}$

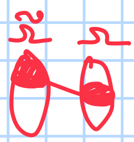
$X^{-1}(A) = \{\omega \in \Omega \mid X(\omega) \in A\}$

① $\tilde{\Omega} \in \tilde{\mathcal{F}}$

② if $B \in \tilde{\mathcal{F}}$ i.e

$B = X^{-1}(A) = \{\omega \in \tilde{\Omega} \mid X(\omega) \in A\}$

$(X^{-1}(A))^c = \{\omega \in \tilde{\Omega} \mid X(\omega) \in A^c\}$



as bijective

$= B^c = X^{-1}(A^c)$

$\therefore B^c \in \tilde{\mathcal{F}}$

③ $B_1, B_2, \dots \in \tilde{\mathcal{F}} \quad B_j^o = X^{-1}(A_j)$

$\bigcup_{j=1}^{\infty} B_j^o = \cup\{\omega : X(\omega) \in A_j\} = \{\omega \in \tilde{\Omega} \mid X(\omega) \in \bigcup_{j=1}^{\infty} A_j\}$

$= X^{-1}(\cup A_j) \Rightarrow X^{-1}(\cup A_j) \in \tilde{\mathcal{F}}$

14. $S \leftarrow$ family of subsets of Ω.

$$\mathcal{F}_S = \cap \{ \mathcal{F} \mid \mathcal{F} \text{ is a sigma field s.t } S \subset \mathcal{F} \}$$

$\mathcal{F}_S$ is a sigma field:

$\Omega \in \mathcal{F}_\alpha \quad , \forall \alpha \in \Lambda$
$\phi \in \mathcal{F}_\alpha , \quad \forall \alpha \in \Lambda$

now if $\quad A \in \mathcal{F}_S \Rightarrow A \in \mathcal{F}_\alpha , \forall \alpha \in \Lambda$
then
as $A^c \in \mathcal{F}_\alpha , \forall \alpha \in \Lambda$
$\Rightarrow A^c \in \mathcal{F}_S$

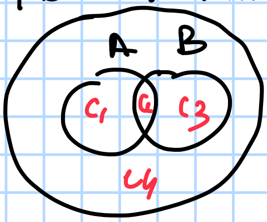
also if $\quad A_1, A_2, \ldots\ldots \in \mathcal{F}_S \Rightarrow A_1, A_2, \ldots \in \mathcal{F}_\alpha , \forall \alpha \in \Lambda$
$\Rightarrow \bigcup_{n=1}^{\infty} A_i \in \mathcal{F}_\alpha , \forall \alpha \in \Lambda$
$\Rightarrow \mathcal{F}_S$ is σ-field.

if $\mathcal{F}_S$ is not the smallest σ-field
then let $\mathcal{F}_\gamma$ be smallest
$|\mathcal{F}_\sigma| < |\mathcal{F}_S|$

and also as $\mathcal{F}_\gamma$ is a σ-field
$\mathcal{F}_S = \ldots \cap \mathcal{F}_\gamma \cdot \ldots$
so $|\mathcal{F}_S| \leq |\mathcal{F}_\gamma| \quad *$
$\therefore \mathcal{F}_S$ is smallest σ-field

or let
$\mathcal{G}$ be σ-field
cont S.
$\mathcal{G} \in \Delta$
$\Rightarrow \mathcal{F}_S \subseteq \mathcal{G}$
$\mathcal{F}_S$ σ-field genrated by S
is smallest

15. A, B containing Ω



Ω

σ-field $\mathcal{F} = \{ \phi, \Omega, A, B, A^c, B^c, A \cup B, A \cap B,$
$A \cup B^c, A \cap B^c,$
$B \cup A^c, B \cap A^c,$
$A^c \cup B^c, A^c \cap B^c \}$

$c_1 \quad c_2 \quad c_3 \quad c_4$

total:
$2^4 = 16$

C_1	C_2	C_3	C_4
✓	✓	✓	✓
✓	✓	✓	✗
✓	✓	✗	✓
✓	✗	✓	✓
✗	✓	✓	✓

✓	✓	✗	✗
✓	✗	✓	✗
✗	✓	✓	✗
✓	✗	✗	✓
✗	✓	✗	✓
✗	✗	✓	✓

✓	✗	✗	✗
✗	✓	✗	✗
✗	✗	✓	✗
✗	✗	✗	✓
✗	✗	✗	✗

16. $\Omega \neq \phi$

$A_1, A_2, \ldots A_n \subseteq \Omega$

$A_i \neq \phi \quad \forall i = 1, 2, \ldots n$

$A_i \cap A_j = \phi \quad \forall i \neq j$

$\bigcup_{i=1}^{n} A_i = \Omega$

$\{A_1, A_2, \ldots A_n\}$ is partition of Ω

$\mathcal{F} = \{\phi, \Omega, A_1, A_2, \ldots A_n, A_1^c, A_2^c, \ldots A_n^c, \ldots\ldots\}$

now $A_1 \cup A_2 \in \mathcal{F}$ then

$(A_1 \cup A_2)^c = A_1^c \cap A_2^c$

$= A_n \cup A_{n-1} \ldots A_3 \in \mathcal{F}$

likewise $\mathcal{F} = \{ \bigcup_{i=1}^{n} A_i^{r_i} \mid r_i = 0 \text{ or } 1 \}$

as if for a particular set of r_i, its converse (others) will also be in it.

as 2 choices of every n.

$\text{card}(\mathcal{F}) = 2^n$

Define $\mathcal{F} = \{ \bigcup_{i \in S} A_i \mid S \subseteq \{1, 2, \ldots n\} \}$

↳ smallest σ-field proof

to show: (i) $\mathcal{F}$ is a sigma field. — $\Omega \in \mathcal{F}$

$B \in \mathcal{F}$

$- B = \bigcup_{i \in S} A_i$

$- B^c = \bigcup_{i \in S^c} A_i \in \mathcal{F}$

- if $B_1, B_2, \ldots \in \mathcal{F}$

$B_j = \bigcup_{i \in S_j} A_i$

$\bigcup B_j = \bigcup_{i \in \bigcup S_j} A_i \in \mathcal{F}$

Now: $\bigcup_{i \in S} A_i \in \mathcal{G} \quad \forall$ σ-algebra $\mathcal{G}$ contains $A_1, \ldots A_j$

$\Rightarrow \mathcal{F} \subseteq \mathcal{G}$

$\text{card } \mathcal{F} = 2^n$

Tutorial-2 :

1.

I: 2w, 3b

II: 3w, 4b

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

$$P(\text{Blue}) = P(\text{Blue}|w)P(w) + P(\text{Blue}|B)P(B)$$

$$= \left(\frac{4}{8}\right)\left(\frac{2}{5}\right) + \left(\frac{5}{8}\right)\left(\frac{3}{5}\right)$$

$$= \frac{8}{40} + \frac{15}{40} = \frac{23}{40}$$

2.

H,T H,T T,T H,H H,H

$$P(\text{Head}) = P(H|C_1)P(C_1) + P(H|C_2)P(C_2) + P(H|C_3)P(C_3) + P(H|C_4)P(C_4) + P(H|C_5)P(C_5)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{5}\right) + (0) + (1)\left(\frac{1}{5}\right) + (1)\left(\frac{1}{5}\right)$$

$$= \frac{3}{5}$$

3. $P(A) \neq 0$

$P(B) \neq 0$

$$P(A|B) \geq P(A) \Leftrightarrow P(B|A) \geq P(B)$$

proof :

$(\Rightarrow)$ $P(A|B)\,P(B) = P(AB)$

$$P(A|B) = \frac{P(AB)}{P(B)} \geq P(A)$$

$$\Rightarrow P(AB) \geq P(A).P(B)$$

$$\Rightarrow \frac{P(BA)}{P(A)} \geq P(B)$$

$$\Rightarrow P(B|A) \geq P(B)$$

$(\Leftarrow)$ trivial

limsup and lim inf in real no of seq^n:

$u_n = \left\{\frac{1}{n}\right\}_{n\geq 1}$ $\quad \{n^2\}_{n\geq 1}$ $\quad \{(-1)^n\}_{n\geq 1}$ → sequences

∴ it may inc, dec, osillation, no pattern at all.

concept of convergence:

$u_n \to u_0$

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t $\forall n \geq n_0$

$|u_n - u_0| < \varepsilon$

Note: convergence ⇒ Bounded

what about Boundedness ⇒ convergence

we need: monotonicity

$\sup U_n = M$

if M is sup U_n then

$\forall n \in \mathbb{N}$

$u_n \leq M$

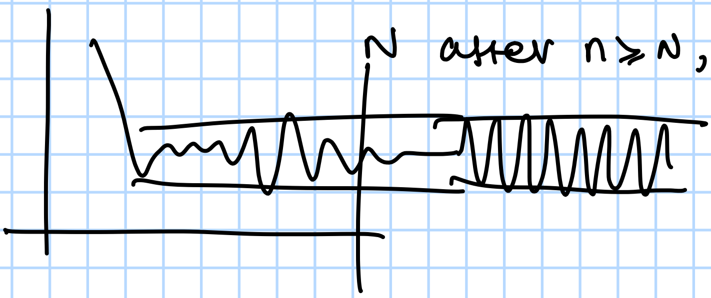
and if $\exists R \in \mathbb{R}$ s.t R is upper bound

then $M \leq R$

$\inf U_n = m$

① $m \leq U_n, \forall n \in \mathbb{N}$

② if r is a lower bound then $r \leq m$



N after $n \geq N$, $\lim \sup U_n \neq \sup U_n$

$\lim \inf U_n \neq \inf U_n$

$V_N = \sup U_K$ for $K \geq N$

$V_1 = 4$

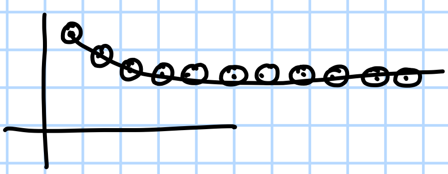
$V_2 = 3$

$V_3 = 2$

$\forall k \geq 4$ $V_K = 1$

∴ after index 4, sup is always 1.

$\{V_N\}_{N\geq 1}$



$\{V_n\}_{N\geq 1}$ is always non-increasing

Now, since $\{V_n\}_{n\geq 1}$ is non-increasing, bounded.

$\{V_n\}_{n\geq 1}$ converges to $\inf V_N$, $N \geq 1$

$= \inf\limits_{N\geq 1} \sup\limits_{K\geq N} U_K$

$= \lim\limits_{n\to\infty} \sup U_n$ (or $\overline{\lim} U_n$)

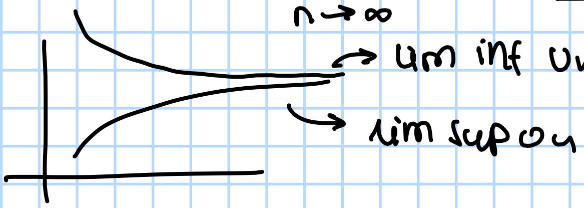
$W_N = \inf\limits_{K\geq N} U_K$ is always non-decreasing

$A \subseteq B$

$\Rightarrow \inf A \geq \inf B$

$\{W_N\} \longrightarrow \sup_{N \geq 1} W_N = \sup_{N \geq 1} \inf_{k \geq N} u_k$

$= \liminf_{n\to\infty} u_n$ (or $\underline{\lim}\, u_n$)



also Note: $v_n \to u$

$\limsup v_n = \liminf u_n = u$

Example: $v_n = (-1)^n\left(1 + \frac{5}{n}\right)$

$V_N = \sup_{k \geq N} U_k = \sup_{k \geq N} \left\{ (-1)^N\left(1+\frac{5}{N}\right), (-1)^{N+1}\left(1+\frac{5}{N+1}\right), \vdots \right\} = \begin{cases} 1+\frac{5}{N} & ; N \text{ is even} \\ 1+\frac{5}{N+1} & ; N \text{ is odd} \end{cases}$

$\limsup v_k = 1$

similarly $\liminf U_k = -1$

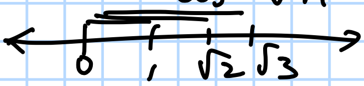
Example: $u_n = \begin{cases} n & ; n \text{ is even} \\ \frac{1}{n} & ; n \text{ is odd} \end{cases}$

$A_n \subseteq A_{n+1}, \; \forall n \in \mathbb{N}$ mon. inc.
$A_n \supseteq A_{n+1}, \; \forall n \in \mathbb{N}$ mon. dec

$A_1 = [0,1]$
$A_2 = [0, \sqrt{2}]$
$\vdots$
$A_n = [0, \sqrt{n}]$

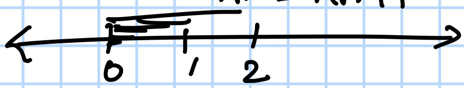
$A_n \subseteq A_{n+1}$

as $\sqrt{n} < \sqrt{n+1}$



$A_n = \left(0, \frac{2}{n}\right]$ $\quad A_1 \supseteq A_2 \cdots$

$A_n \supseteq A_{n+1}$



$\bigcup_{n=1}^{\infty} A_n = \lim_{n\to\infty} A_n$ when $A_n \uparrow n$

$\bigcap_{n=1}^{\infty} A_n = \lim_{n\to\infty} A_n$, when $A_n \downarrow n$

$\bigcup_{n=1}^{\infty} [0, \sqrt{n}] = [0, \infty)$

$= \mathbb{R}^+ \cup \{0\}$

$\bigcap_{n=1}^{\infty} \left[0, \frac{2}{n}\right) = \{0\}$

$C_1 = A_1 \cap A_2 \cap A_3 \cdots \subseteq A_1 \subseteq A_1 \cup A_2 \cup \cdots = B_1$
$C_2 = A_2 \cap A_3 \cap \cdots \subseteq A_2 \subseteq A_2 \cup A_3 \cup \cdots = B_2$
$C_3 = A_3 \cap A_4 \cap \cdots \subseteq A_3 \subseteq A_3 \cup A_4 \cup \cdots = B_3$
$\vdots$

now $C_1 \subseteq C_2 \subseteq C_3 \cdots \quad C_n \uparrow n$
$B_1 \supseteq B_2 \supseteq B_3 \cdots \quad B_n \downarrow n$

$\lim_{n\to\infty} C_n = \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \liminf_{n\to\infty} A_n$

$\lim_{n\to\infty} B_n = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \limsup_{n\to\infty} A_n$

Remark : For monotone seq of sets $\{A_n\}_{n\geq 1} \in \mathcal{F}$. $\lim_{n\to\infty} A_n$ always conv.

proof :

case I : Suppose $A_n \subseteq A_{n+1}$, $\forall n \geq 1$

we say that limit $\lim_{n\to\infty} A_n$ exist if $\lim_{n\to\infty} A_n = \overline{\lim} A_n = \underline{\lim} A_n$

$$\overline{\lim} A_n = \bigcap_{n\geq 1} \bigcup_{k\geq 1} A_k = \bigcap_{n\geq 1} \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} A_k$$

$$\underline{\lim} A_n = \bigcup \bigcap A_k = \bigcup_{k=1}^{\infty} A_k \text{ for } A_n \uparrow n$$

Case II : $A_n \supseteq A_{n+1}$, $\forall n \geq 1$ (Try)

Example 1 : $A_n = (-1/n, 1-1/n]$, $\forall n \geq 1$

$$B_n = \bigcup_{k\geq n} A_k = \left(-\frac{1}{n}, 1\right)$$

$$\lim\sup A_n = \bigcap_{n\geq 1} B_n = [0, 1)$$

Example 2 : $A_n = \{0\} \cup \{\frac{1}{k} \mid k \geq n\}$ $n \geq 1$

find $\lim A_n$

4. A event
$P(A\cap B) = P(A).\,P(B)$ → definition of ind events.
$\forall B \in \mathcal{F}$ $\quad P(A\cap B) = P(A)\,P(B)$

now for $B = A \in \mathcal{F}$
$P(A\cap A) = P(A)^2$
$P(A) = P(A)^2$ $\quad$ here $P(A) = P(A)^2$
$\Rightarrow P(A) = 1$ $\quad \Rightarrow P(A) = 0$ or 1
as $P(A) \geq 0$ $\quad$ not only 0.

6. $P(A\cup B) = P(A) + P(B) - P(A\cap B) \leq 1$
$P(A) + P(B) - 1 \leq P(A\cap B)$

now
if $P(\bigcap_{i=1}^{k} A_i) \geq \sum_{i=1}^{k} P(A_i) - (k-1)$

$P(\bigcap_{i=1} A_i \cap A_{k+1}) \geq P(\bigcap_{i=1}^{k} A_i) + P(A_{k+1})$

use induction
(called Bonferroni ineq)

$\geq \sum_{i=1}^{k+1-1} P(A_i) - (k-1) - 1$

$= \sum_{i=1}^{k+1} P(A_i) - (k+1-1)$

$\therefore P(\bigcap_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$

7. $P(A\cup B) = P(A) + P(B) - P(A\cap B) \leq P(A) + P(B)$
for $k=2$ true
if for $k=k$ true $\quad$ (countable subadditivity)
$P(\bigcup_{i=1}^{k} A_i) \leq \sum_{i=1}^{k} P(A_i)$ $\quad$ (Boole's ineq)

$P(\bigcup_{k=1}^{n} A_k) \leq \sum_{k=1}^{n} P(A_k)$
$\leq \sum_{k=1}^{\infty} P(A_k)$

for $k' = k+1$

$P(\bigcup_{i=1}^{k+1} A_i) \leq P(\bigcup_{i=1}^{k} A_i) + P(A_{k+1})$ $\quad$ suppose $C_n = \bigcup_{k=1}^{n} A_k$

$B_1 = A_1$, $B_n = C_n \setminus C_{n-1}$ $\forall n \geq 2$

$P(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(\bigcup_{k=1}^{n} A_k)$ $\quad \leq \sum_{i=1}^{k} P(A_i) + P(A_{k+1})$ $\quad B_2 = A_2 \cap A_1^c$
$B_3 = A_3 \cap A_2^c \cap A_1^c$

$\leq \lim_{n\to\infty} \sum_{k=1}^{n} P(A_k) \leq$ $\quad = \sum_{i=1}^{k+1} P(A_i)$ $\quad$ (This is for the finite case) $\quad B_n \in \mathcal{F}$

$\sum_{k=1}^{\infty} P(A_k)$ $\quad \therefore$ true for n $\quad C_n = \bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k$ ← This is true
and $B_k \subseteq A_k$, $\forall k \geq 1$

8. now as $P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$ $\quad P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_n)$
$= \sum_{k=1}^{\infty} P(B_k)$

and $P(A_i) = 0$, $\forall i \in \mathbb{N}$ $\quad = \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k)$

$\sum_{i=1}^{\infty} P(A_i) = 0$ $\quad = \lim_{n\to\infty} P(\bigcup_{k=1}^{n} B_k)$

$\Rightarrow P(\bigcup_{i=1}^{\infty} A_i) \leq 0$, also as $P \geq 0$ $\quad = \lim_{n\to\infty} P(\bigcup_{k=1}^{n} A_k)$

$$\Rightarrow \quad P\left(\bigcup_{i=1}^{\infty} A_i\right) = 0$$

9. $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$ (Already proved)

now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

for $n=2$ (true)

let it be true for $n=k$

then $\sum_{i=1}^{K} P(A_i) - \sum_{i>j} P(A_i \cap A_j) \leq P\left(\bigcup_{i=1}^{K} A_i\right)$

for $n = k+1$

$$P\left(\bigcup_{i=1}^{K} A_K \cup A_{k+1}\right) = P\left(\bigcup_{i=1}^{K} A_i\right) + P(A_{K+1}) - P\left(\left(\bigcup_{i=1}^{K} A_K\right) \cap A_{k+1}\right)$$

$$\geq \sum_{i=1}^{K} P(A_i) - \sum_{i>j}^{K} P(A_i \cap A_j) + P(A_{K+1}) - P\left(\left(\bigcup_{i=1}^{K} A_K\right) \cap A_{K+1}\right)$$

$$= \sum_{i=1}^{K+1} P(A_i) - \sum_{i>j}^{K} P(A_i \cap A_j) - P\left(\left(\bigcup_{i=1}^{K} A_K\right) \cap A_{K+1}\right)$$

__Note__: $P((A_1 \cap A_{K+1}) \cup (A_2 \cap A_{K+1}) \cdots \cup (A_K \cap A_{K+1}))$

$$\leq \sum_{i=1}^{K} P(A_i \cap A_{K+1})$$

$$\Rightarrow -P\left(\left(\bigcup_{i=1}^{K} A_i\right) \cap A_{K+1}\right) \geq -\sum_{i=1}^{k} P(A_i \cap A_{k+1})$$

then $$P\left(\bigcup_{i=1}^{K+1} A_i\right) \geq \sum_{i=1}^{K+1} P(A_i) - \sum_{i>j}^{K} P(A_i \cap A_j) - \sum_{i=1}^{K} P(A_i \cap A_{K+1})$$

$$= \sum_{i=1}^{K+1} P(A_i) - \sum_{i>j}^{K+1} P(A_i \cap A_j)$$

5. $$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

$$= (A_1 \cup A_2 \ldots) \cap (A_2 \cup A_3 \ldots) \cap (A_3 \cup A_4 \ldots) \ldots$$

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

$$= (A_1 \cap A_2 \ldots) \cup (A_2 \cap A_3 \ldots) \cup (A_3 \cap A_4 \ldots) \ldots$$

$\{A_n\} \longrightarrow$ limit A if : $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$

(a) as $\forall k \in \mathbb{N}$
$A_k \in \mathcal{F}$
also $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$ (property)

then $\bigcup_{k=n}^{\infty} A_k \in \mathcal{F}$ $\forall n \in \mathbb{N}$
as countable union in $\mathcal{F}$

As $A_n \in \mathcal{F}$
$\bigcup_{k\geq 1}^{\infty} A_k \in \mathcal{F}$
$\bigcup_{k\geq 1}^{n-1} \in \mathcal{F}$
$\bigcup_{k\geq 1}^{\infty} A_k \setminus \bigcup_{k\geq 1}^{n-1} A_k = \bigcup_{k\geq n} A_k \in \mathcal{F}$

then intersection (countable) $\in \mathcal{F}$
(same for others)

(b) for $\limsup_{n\to\infty} A_n = \omega$
then
as $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$
$\omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$
$\Rightarrow \omega \in \bigcup_{k=1}^{\infty} A_k, \bigcup_{k=2}^{\infty} A_k, \ldots\ldots$
$\Rightarrow$ as $\omega \in \bigcup_{k=1}^{\infty} A_k, \bigcup_{k=2}^{\infty} A_k, \bigcup_{k=3}^{\infty} A_k, \ldots$

$\omega \in A_n$ for finite n
then for $\bigcup_{k=n+1}^{\infty} A_k \not\ni \omega$ $\ast$
$\therefore$ $\omega \in A_n$ for infinite many n.

$\omega \in A_n$ for all but fin. many n mean that $\omega \in A_n$ as $n\to\infty$.

Suppose $\omega \in \underline{\lim} A_n = \bigcup_{n\geq 1} \bigcap_{k\geq n} A_k$
$= \bigcup_{n\geq 1} C_n$ say
$\Leftrightarrow \exists$ some $n_1 \in \mathbb{N}$ s.t
$\omega \in C_{n_1} = \bigcap_{k\geq n_1} A_k$
$\Leftrightarrow \omega \in A_k, \forall k \geq n_1$
$\Leftrightarrow \omega \in A_n$ for all but finitely many n or $\omega \in A_n$ eventually.

for $\liminf_{n\to\infty} A_n = \bigcup_{k=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \omega$

then $\omega \in \bigcap_{k=1}^{\infty} A_k$ or $\bigcap_{k=2}^{\infty} A_k$ or ....

$\Rightarrow \omega \in A_i, \forall i \in \mathbb{N}$ or
$A_i, \forall i \in \mathbb{N}\setminus\{1\}$ or
$A_i, \forall i \in \mathbb{N}\setminus\{1,2\}$ or
...

i.o = infinitely often

so $\exists n \in \mathbb{N}$ s.t
$\omega \in A_i, \forall i \in \mathbb{N}\setminus\{1,2,\ldots n\}$

$\therefore \liminf_{n\to\infty} A_n = \{\omega \mid \omega \in A_n$, for all but finitely many $n\}$

suppose $\omega \in \overline{\lim} A_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k$
$= \bigcap_{n\geq 1} B_n$ (say)
$\Leftrightarrow \omega \in B_n, \forall n \geq 1$
$= \bigcup_{k\geq n} A_k$
suppose $n=1$, $\omega \in \bigcup_{k\geq 1} A_k$
$\Leftrightarrow \exists$ some $n_1 \in \mathbb{N}$
$\omega \in A_{n_1}$
choose $n = n_1+1$, $\omega \in \bigcup_{k\geq n_1+1} A_k$
$\Leftrightarrow \exists$ some $n_2 > n_1$
$\ni \omega \in A_{n_2}$
$\vdots$

(c) let $x \in \liminf_{n\to\infty} A_n$, then
$\exists n \in \mathbb{N}$ s.t
$x \in A_i, \forall i \in \mathbb{N}\setminus\{1,2,\ldots n\}$
$\Rightarrow x \in \bigcup_{i=1}^{\infty} A_i, x \in \bigcup_{i=2}^{\infty} A_i, \ldots\ldots x \in \bigcup_{i=n}^{\infty} A_i, \forall n \in \mathbb{N}$
$\Rightarrow x \in \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k = \limsup_{n\to\infty} A_n$

(d) $\left(\bigcup_{i=1}^{\infty}\left(\bigcap_{k=i}^{\infty} A_k\right)\right)^c = \bigcap_{i=1}^{\infty}\left(\bigcup_{k=i}^{\infty} A_k^c\right)$ (from tut 1)

$\underset{A}{\shortparallel}$ $= \lim_{n\to\infty} \sup A_k^c$

now as A, A^c are disjoint i.e

$$A \cap A^c = \phi$$
and $$A \cup A^c = \Omega$$

$$P(A \cup A^c) = P(\Omega) = 1$$
$$= P(A) + P(A^c) - P(A \cap A^c)$$
$$= P(A) + P(A^c) - 0$$

$$P(A) = 1 - P(A^c)$$
$$\therefore P(\lim_{n\to\infty} \inf A_n) = 1 - P(\lim_{n\to\infty} \sup A^c)$$

10. $P(\lim_{n\to\infty} \inf A_n) \leq \underbrace{\lim_{n\to\infty} \inf P(A_n) \leq \lim_{n\to\infty} \sup P(A_n)}_{\text{trivial}} \leq P(\lim_{n\to\infty} \sup A_n)$

Denote $C_n = \bigcup_{k\geq n} A_k$ $\downarrow n$, $B_n = \bigcap_{k\geq n} A_k$, $\uparrow n$

now, $P(\lim \inf A_n) = P(\bigcup_{n\geq 1}^{\infty} B_n) = P(\lim_{n\to\infty} B_n)$

$$= \lim_{n\to\infty} P(B_n)$$
$$= \lim_{n\to\infty} \inf P(B_n) \leq \lim_{n\to\infty} \inf P(A_n)$$

$P(\lim \sup A_n) = P(\bigcap_{n\geq 1} C_n) = P(\lim_{n\to\infty} C_n) = \lim_{n\to\infty} P(C_n) = \lim_{n\to\infty} \sup P(C_n)$

$$\geq \lim_{n\to\infty} \sup P(A_n)$$

11. $A_n \to A$ then

$\lim_{n\to\infty} \sup A_n = \lim_{n\to\infty} \inf A_n$

then $\Rightarrow \lim_{n\to\infty} \inf P(A_n) = \lim_{n\to\infty} \sup P(A_n)$ And sandwich theorm

$\Rightarrow P(A_n) \to P(A)$

(defination)

12. $\sum_{n=1}^{\infty} P(A_n) < \infty$

$P(\bigcap_{i=1}^{n} \bigcup_{k=i}^{n} A_k) \leq P(\bigcup_{k=1}^{n} A_k)$

doubt

Borel cantelli lemma

$P(\lim \sup A_n) = P(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n}^{\infty} A_n) = \lim_{n\to\infty} P(\bigcup_{k\geq n} A_k)$

$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k) = 0$

(why)

tail series

13. A is event $P(A) = P(A|B)$

this means

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = P(A)$$

or $P(A\cap B) = P(A)P(B)$

for $B = A$

$P(A) = 1$

as $B \neq \phi$

14. A_1, A_2, A_3 are ind

$\Rightarrow P(A_1)P(A_2) = P(A_1\cap A_2)$

and

$P(A_1^c)P(A_2) = (1-P(A_1))P(A_2)$

$= P(A_2) - P(A_1\cap A_2)$

$= P(A_2 - A_1\cap A_2)$

$= P(A_2 \cap A_1^c)$

$\Rightarrow A_1^c, A_2, A_3$ are ind

similarly

$\Rightarrow A_1^c, A_2^c, A_3$ are ind

similarly

$\Rightarrow A_1^c, A_2^c, A_3^c$ are ind

$\Rightarrow$ ⋮

$\Rightarrow$ ⋮

$\Rightarrow A_1, A_2, A_3$ are ind.

<u>Note</u>: This notion holds for infinite subsets too

(a) $\Rightarrow$ (b) A_1, A_2, A_3 are ind.

$\Rightarrow P(A_i\cap A_j) = P(A_i)P(A_j)$

and $P(\bigcap_{k=1}^{3} A_k) = \prod_{k=1}^{3} P(A_k)$

<u>claim</u>: A_1^c, A_2, A_3 are ind

for $j = 2, 3$

$P(A_1^c \cap A_j) = P(A_j)(1-P(A_1))$

<u>Note</u>: $\sigma(A) = \bigcap_{\mathcal{J}_i \supset A} \mathcal{J}_i$

$\sigma(X) = \sigma\{X^{-1}(B) \mid B\in \mathcal{B}(\mathbb{R})\} = A$

sigma field by X

$F_{XY}(x,y) = F_X(x)F_Y(y)$

$\sigma(X_1, X_2)$

15. fair dice rolled

$$\{(1,1), (1,2), \dots (1,6),$$
$$(2,1), (2,2), \dots (2,6),$$
$$\vdots \qquad \vdots$$
$$(6,6)\}$$

$\{(1,6), (2,5), \dots (6,1)\} = E = B$

$P(E) = \frac{6}{36} = \frac{1}{6} = P(B)$

$P(A|B) = 1/6$ for every digit

$B \rightarrow$ sum is 7

$A \rightarrow$ score shown by 1st dice

$= P(A)$

$A \rightarrow$ score by 1st

$\therefore P(A|B) = P(A)$

16. fair die n times

A_{ij} event that

i^{th} and j^{th} roll produce same number

Pairwise A_{ij}, $B_{i'j'}$ then

i^{th} j^{th}

$\underbrace{\quad}_{n \text{ times}}$

$1 \leq i < j \leq n$

$P(A_{ij}) = P$ that i^{th} and j^{th}

case I : suppose (i,j) and (i,j') with $i<j<j'$
i.e A_{ij} $A_{ij'}$

$P(A_{ij}) = \frac{1}{1} \times \frac{1}{6} = \frac{1}{6}$

$P(B_{i'j'}) = \frac{1}{6}$

same
now $n-2$ places remain
Any values $= 6^{n-2}$
so $6^{n-2} \times 6$ total cases

$P(A_{ij}) = \frac{6^{n-2} \times 6}{6^n}$

$P(A_{ij}) = \frac{1}{6}$

$P(A_{ij} A_{ij'}) = \frac{6^{n-3} \times 6}{6^n} = \frac{1}{36}$

$P(A_{ij}) P(A_{ij'}) = \frac{1}{36}$

$P(A_{ij} | B_{i'j'}) = P(i=i') P(j \neq j') + P(i=i') P(j=j') + P(i \neq i') P(j=j') + P(i \neq i') P(j \neq j')$

case II : $i<j$
$i'<j'$
not all same

$\frac{6^{n-4} \times 6 \times 6}{6^n} = \frac{1}{36}$ $\therefore$ independent

$\rightarrow P(A_{12} \cap A_{23} \cap A_{31})$
same

but $\frac{6 \times 6^{n-3}}{6^n} = \frac{6^{n-2}}{6^n} = \frac{1}{6^2}$

$P(A_{12}) P(A_{23}) P(A_{31}) = \frac{1}{6^3}$

17. fair coin : tossed 3 times

A = ○ ○ ○
diff

B = ○ ○ ○
diff

C = ○ ○ ○
diff

$A = \{HTH, HTT, THH, THT\}$

$B = \{HTH, TTH, HHT, THT\}$

$C = \{HHT, HTT, THH, TTH\}$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{HTH, THT\})}{P(B)} = \frac{1}{2}$

$P(A) = \frac{1}{2} = P(A|B) = P(A|C)$

$P(A | B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{0}{P(B \cap C)} = 0 \neq P(A)$

$\therefore$ A is ind of B and C does not mean A is ind of $B \cap C$.

Tutorial-3:

1. $\mathcal{F} \subseteq \mathcal{G}$ X is a random variable
i.e $\forall x \in \mathbb{R}$
$\{\omega \mid X(\omega) \leq x\} \in \mathcal{F}$
as $\mathcal{F} \subseteq \mathcal{G}$
$\Rightarrow \{\omega \mid X(\omega) \leq x\} \in \mathcal{G}$

$\mathcal{F} \subset \mathcal{G}$
see Borel function
$(\Omega, \mathcal{F}, P)$ $P: \mathcal{F} \to [0,1]$
set function

$\mathbb{R}: (a,b)$
$\{X(\omega) \leq x\} \in \mathcal{F}$ ⤳ $\mathcal{F}$ generated by (a,b)
$\Rightarrow X^{-1}(-\infty, x] \in \mathcal{F}$
i.e $X^{-1}(B) \in \mathcal{F}$
$\forall B \in \mathcal{B}(\mathbb{R})$
$\mathcal{B}(\mathbb{R}) = \sigma\{(a,b) \mid a < b, a,b \in \mathbb{R}\}$
$\{\Omega, \mathcal{F}, P\} \xrightarrow{X} \{X(\Omega), \mathcal{B}(\mathbb{R}), P\circ X^{-1}\}$ P_X
$\{\mathbb{R}, \mathcal{B}(\mathbb{R}), P\circ X^{-1}(-\infty, x]\} = P[\{X(\omega) \leq x\}]$
that is why in def of X, $\{X(\omega) \leq x\} \in \mathcal{F}$

2. $\Omega = \{-2,-1,0,1,2\}$
$X(\omega) = |\omega|$
$X(\omega) = 2\omega + 1$
smallest sigma field

$X(\omega)$: $2 - \{-2,2\}$, $1 - \{-1,1\}$, $0 - \{0\}$

from this $X(\omega)$ we have
$\mathcal{F} = \{ \{-2,2\}^{c_1} \cup \{-1,1\}^{c_2} \cup \{0\}^{c_3} \}$ for $c_1, c_2, c_3 = 0$ or 1
($B_1 = \{-2,2\}$, $B_2 = \{-1,1\}$, $B_3 = \{0\}$)

$X(\omega)$: $-3 - \{-2\}$, $-1 - \{-1\}$, $1 - \{0\}$, $3 - \{1\}$, $5 - \{2\}$

$\mathcal{F} = \{ \{-2\}^{c} \cup \{-1\}^{c} \cup \{0\}^{c} \cup \{1\}^{c} \cup \{2\}^{c} \}$

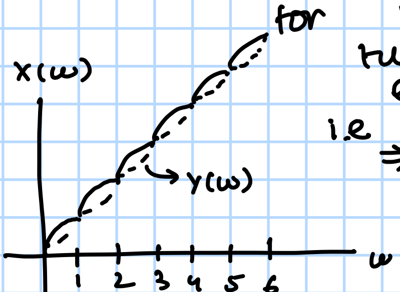
$\mathcal{F}_1 = \{ \bigcup_{i \in J} B_i \mid J \subseteq \{1,2,3\} \}$

$$X^{-1}(B) = X^{-1}(-\infty, x] = \begin{cases} \phi & x < 0 \\ \{0\} & 0 \leq x < 1 \\ \{-1,0,1\} & 1 \leq x < 2 \\ \{-2,-1,0,1,2\} & 2 \leq x \end{cases}$$

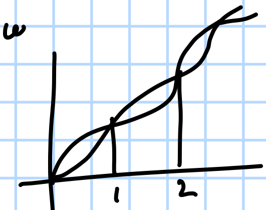
$X^{-1}(B) \in \mathcal{F} \subset \mathcal{G}$
$\forall B \in \mathcal{B}(\mathbb{R})$
$\Rightarrow X^{-1}(B) \in \mathcal{G}$
$\forall B \in \mathcal{B}(\mathbb{R})$

3. $F_X = F_Y$
now
$F(x) = P(X \leq x)$
and $F(Y) = P(Y \leq y)$

for $F_X = F_Y$
they have to be equal at every $r \in \mathbb{R}$
i.e $F_X(r) = F_Y(r)$ $\forall r \in \mathbb{R}$
$\Rightarrow P(X \leq r) = P(Y \leq r)$
but $X(\omega) \neq Y(\omega)$



$0.2\sin(\omega\pi) + \omega$
$-0.2\sin(\omega\pi) + \omega$



here when
$\Omega = \{1, 2, \ldots 6\}$
$\mathcal{F} = P(\Omega)$
and $P(\{i\}) = \frac{1}{6}$

$X \sim$ Bernoulli $(p = 1/2)$
$$P(X = x) = \begin{cases} 1/2 & x = 1 \\ 1/2 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$
$Y = 1 - X \sim$ Bernoulli $(p = 1/2)$
$P(X = 1 - X) = P(X = \frac{1}{2})$

measure
$f_X(x)$: pmf of X
$\mu: \mathcal{F} \to [0, \infty)$
a) $\mu(\phi) = 0$
b) $\mu(\cdot) \geq 0$

from the function we see that $\neq 1$
$X(\omega) = 0.2\sin(\omega\pi) + \omega$
$Y(\omega) = -0.2\sin(\omega\pi) + \omega$
then if $r \in (n, n+1)$
then this values is also b/w $(n, n+1)$
$\therefore \{\omega \mid X(\omega) \leq r\} = \{\omega \mid Y(\omega) \leq r\}$
$\therefore P(X \leq r) = P(Y \leq r)$
same for $r = n$ case

eg $A \in \mathcal{F}$, $\mu(A) = |A|$ card of A
called counting measure
$\mu(A) =$ length of A ← length of line, area, etc.

$P\circ X^{-1} = \frac{dP}{dC}$ ← discrete
dC → counting measure
$\frac{dP}{dL} = P\circ X^{-1}$ ← cont

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$
$\int f(x)\,dx = 1$
* as area under curve $= 1$

4. (a) $X(\omega) = 1$
$\omega \in A$
$X(\omega) = 2$ ← otherwise
$P(A) = \frac{1}{4}$

$$\text{now } \{\omega \mid X(\omega) \leq x\} = \begin{cases} \phi & ; x \in (-\infty, 1) \\ A & ; x \in [1, 2) \\ \Omega & ; x \in [2, \infty) \end{cases}$$

$$F(x) = \begin{cases} 0 & ; x \in (-\infty, 1) \\ 1/4 & ; x \in [1, 2) \\ 1 & ; x \in [2, \infty) \end{cases}$$

$X \sim$ uniform $(0,1)$
$$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) $X(\omega) = \begin{cases} \phi & ; x \in (-\infty, -1) \\ B & ; x \in [-1, 1) \\ A \cup B & ; x \in [1, 2) \\ \Omega & ; x \in [2, \infty) \end{cases}$

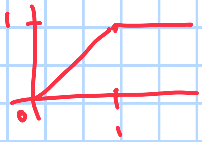
given $A \cap B = \phi$

$\therefore P(A \cup B) = P(A) + P(B)$

now

$$F(x) = \begin{cases} 0 & ; x \in (-\infty, -1) \\ 1/2 & ; x \in [-1, 1) \\ 3/4 & ; x \in [1, 2) \\ 1 & ; x \in [2, \infty) \end{cases}$$

$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$



$F_X(x) = P(X \leq x)$
$= \int_0^x f(t)\,dt$
$= \int_0^x dt = x$

$F_Y(x) = P(Y \leq x)$
$= P(1 - X \leq x)$
$= P(X \geq 1 - x)$

(c) $F(x) = \begin{cases} 0 & ; x \in (-\infty, c_1) \\ \alpha_1 & ; x \in [c_1, c_2) \\ \alpha_1 + \alpha_2 & ; x \in [c_2, c_3) \\ \vdots & \vdots \\ \alpha_1 + \alpha_2 + \dots \alpha_{n-1} & ; x \in [c_{n-1}, c_n) \\ 1 & ; x \in [c_n, \infty) \end{cases}$

5. (a) $F(x) = \frac{1}{2}$ for $x \in \mathbb{R}$

Not a distribution function

as $F(x) = P(X \leq x)$

and as X is a random variable

for $x \to \infty$

$\{\omega \mid X(\omega) \leq x\} = \Omega$

as $X : \Omega \to \mathbb{R}$

and $P(X \leq x)$ is an increasing function, for $x \to \infty$

$P(X \leq x) = 1$

we $\forall x \in \mathbb{R}$ $F(x) = \frac{1}{2}$, not a dist. function.

(b) $F(x) = 0$ for $x \leq 0$

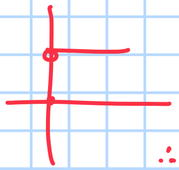
$F(x) = 1$ for $x > 1$

doubt

as if $x > 1$ then

can/cannot be, depends

$x > 0$ $\therefore$

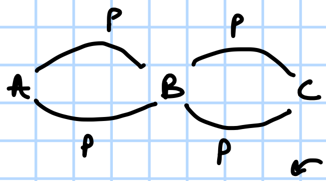


Not a right cont function

$\therefore$ Not $F_X(x)$

$\lim_{x \to 0^+} F_X(x) = 1 \neq 0 = F_X(0)$

6.



total options: $2 \times 2 = 4$

$\leftarrow$ Prob that AB is open

$P(AB) = P(AB_1 \cup AB_2)$
$= P(AB_1) + P(AB_2) - P(AB_1 \cap AB_2)$
$= (1-p) + (1-p) - P(AB_1) P(AB_2)$ ($\because$ independent)
$= (1-p) + (1-p) - (1-p)^2$

$P(AB \cap BC) = P(AB) P(BC)$
$= [2 - 2p - (1-p)^2]^2$
$= [2 - \cancel{2p} - (1 + p^2 - \cancel{2p})]^2$
$= [2 - 1 - p^2]^2$
$= [1 - p^2]^2$

7. one of r
i^{th} outcome $P(\{i\}) = p_i$
$\sum p_i = 1$

$x_1 + x_2 + \cdots x_r = n$ — ①
also, n times any x_1 times P_1
x_2 times P_2
$\vdots$
x_r times P_r
then probability $= \underbrace{\dfrac{(n)!}{(x_1)!(x_2)!\cdots(x_r)!}}_{\text{different ways of permuting}} \times (P_1)^{x_1} \times (P_2)^{x_2} \times \cdots (P_r)^{x_r}$

8. 52 seats for 50 seats
5% people don't show up

$P(51 \text{ people show up}) = ({}^{52}C_1)(0.05)(0.95)^{51}$
$P(52 \text{ people show up}) = (0.95)^{52}$

$P(1,2,3,\ldots 50 \text{ people show up}) = 1 - P(51) - P(52)$

9. low end tv – 50%
high end – 20%
browsing – 30%
5 customers into 3 categories
20% 2, 50% 1, 30% 2 $\quad \dfrac{5!}{2!.2!.1!} \leftarrow$ dividing into 3 categories

every category $(0.2)^2(0.5)(0.3)^2\left(\dfrac{5!}{2!.2!.1!}\right)$

10. n, P 0 to n
binomial
$P(X=k) = \binom{n}{k}(p)^k(1-p)^{n-k}$

now $P(X=0) = \binom{n}{0}(p)^0(1-p)^n$
$P(X=1) = \binom{n}{1}(p)^1(1-p)^{n-1}$

if $P(X=k) \leq P(x=k+1)$
then $\binom{n}{k} p^k(1-p)^{n-k} \leq \binom{n}{k+1} p^{k+1}(1-p)^{n-k-1}$

$\dfrac{\cancel{n!}}{k!(n-k)!}\cancel{p^k}\,\cancel{(1-p)^{n-k}} \leq \dfrac{\cancel{n!}}{(k+1)!(n-k-1)!}\cancel{p^k}.\dfrac{p\,\cancel{(1-p)^{n-k}}}{(1-p)}$

$\dfrac{1-p}{n-k} \leq \dfrac{p}{k+1}$ given $p \in (0,1)$
$k - \cancel{kp} + 1 - p \leq np - \cancel{pk}$
$\Rightarrow k + 1 - p \leq np$
$\Rightarrow k \leq np + p - 1$
$\Rightarrow k \leq p(n+1) - 1$
or till $\lfloor p(n+1) \rfloor - 1$ k will

be s.t $P(X=k) \leq P(X=k+1)$

∴ for $k = \lfloor p(n+1) \rfloor$ largest value

11. $\lambda = 1$ P(atleast 1 error)

$$= P(X=1) + P(X=2) + \dots$$
$$= \sum (1)^k \frac{(e)^{-\lambda}}{k!}$$
$$= \sum \frac{e^{-1}}{k!} = \frac{1}{e}\left(\sum \frac{1}{k!}\right)$$
$$= \frac{1}{e}\left(e^1 - 1\right)$$
$$= 1 - \frac{1}{e}$$

← counting the no. of failures

if $P(X=x)$ ← x may failures

$(1-p)^x p$

trial model

$x = 0, 1, \dots$

failure model

$Y \overset{d}{=} 1 + X$

12. $P(X=Y)$

↳ if $X = k$

then $Y = k$

$$P(X=Y) = \sum (1-p)^{k-1}p \; (1-p)^{k-1}p$$
$$= \sum \left((1-p)^{k-1}p\right)^2$$

doubt, ask what notation means.

13. same doubt

what is the meaning.

14. Now Note:

$P(X=k)$

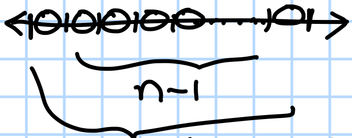
then k trials needed before any outcome occured at least once

∴ $p_1^{r_1} p_2^{r_2} \dots p_m^{r_m}$

s.t $r_1 + r_2 \dots r_m = n$

and also atleast one is placed last

wlog $r_1 + r_2 + \dots r_{m-1} = n-1$



n-1, also select m spaces

So $\binom{n}{m}$ ← no of ways of making colums

also now $\binom{n}{m} \times \frac{(m-1)!}{(r_1)!(r_2)!\dots(r_{m-1})!}$ different permutations of things

problems: let $A_1, A_2, \dots A_n$ and B be independent events.

$\mathcal{F}_{\{A_1, \dots A_n\}}$: σ generated by $\{A_1, \dots A_n\}$

to show − B is ind of C for any $C \in \mathcal{F}_{\{A_1, \dots A_n\}}$

12. $P(X=Y)$

$$= \sum_{k=1}^{\infty} (1-p)^{k-1}(p)(1-p)^{k-1}(p) = \frac{p^2}{(1-p)^2}\sum_{k=1}^{\infty}(1-p)^{2i}$$

$$= \frac{p^2}{(1-p)^2} \times \frac{(1-p)^2}{1-(1-p)^2}$$

$$= \frac{p\cancel{^2}}{\cancel{(1-p)^2}} \times \frac{\cancel{(1-p)^2}}{(\cancel{1-1+p})(1+1-p)}$$

$$= \frac{p}{2-p}$$

$$P(X>Y) = \sum_{i=1}^{\infty}\sum_{j=1}^{i-1} P(X=i, Y=j)$$

$P(X<Y)$ same

$$P(X=k \mid X>Y) = \frac{P(X=k, X>Y)}{P(X>Y)} = P(X=k)\frac{P(Y<K)}{P(Y<X)}$$

$$P(X+Y=l) = (l-1)p^2(1-p)^{l-2}$$

$$= \sum_{x=1}^{l-1} p(1-p)^{x-1} p(1-p)^{l-x-1}$$

$$= \sum_{x=1}^{l-1} p(1-p)^{\cancel{x}-1} p(1-p)^{l-\cancel{x}-1}$$

$$= \sum_{x=1}^{l-1} p^2(1-p)^{l-2} = (l-1)p^2(1-p)^{l-2}$$

13. $F_U(U) = P(U \leq u)$

$= P(\max(X,Y) \leq U)$

$= P(X \leq U, \; Y \leq U)$

$= P(X \leq U)\, P(Y \leq U)$

$P(V \leq u) = P(\min(X,Y) \leq u)$

$= 1 - P(\max(X,Y) > U)$

$= 1 - P(X>u)\, P(Y>V)$

14. $P(X=n)$ $n = m, m+1, \ldots$

$P(X>n)$ A_i – event that ith outcome has not occured at least once wrt to nth trial.

$$P\left(\bigcup_{i=1}^{m} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i_1<i_2} P(A_{i_1} \cap A_{i_2}) + \cdots (-1)^{m+1} \underbrace{P(A_1 \cap A_2 \cap \cdots A_m)}_{0}$$

$P(A_i) = (1-p_i)^n$

$P(A_{i_1} \cap A_{i_2}) = (1-p_{i_1} - p_{i_2})^n$

$P(X=n) = P(X \geq n-1) - P(X>n)$

problem set - 4 -

1. X = no of acidents on higway

This is poission distribuition

$$E(X) = \lambda = 3$$
or $\lambda = 3$

Now, $P(X=0) = \frac{e^{-\lambda}\lambda^{(0)}}{0!}$

$$= e^{-3}\frac{(1)}{1} = \frac{1}{e^3}$$

$$P(X=0) = \frac{1}{e^3}$$

X: # of acidents occured among given n many acicents -
let p be prob of ocurng an incident
$X \sim bin(n, p)$
when n is larger, p is suff small
$X \sim Poi(\lambda)$
$\lambda = E(X) = 3$ (given)

2. (a) total possible ways n many balls can be dist in n boxes $= n^n$

Since 1 box $\binom{n}{1}$ way, (n-1) will have exactly 2 balls so $\binom{n-1}{1}$ ways

choose 2 balls in $\binom{n}{2}$ ways

no of ways = choosing 1 box for 0 balls x choosing 1 box for 2 balls x choo's 2 balls x others

$$\text{prob} = \binom{n}{1}\binom{n-1}{1}\binom{n}{2}(n-2)! \,/\, (n)^n$$

(b) box 1 is empty only 1 box is empty prob = eqy obv box atlst once

Total $= (n-1)^n$ → n-1 boxes for each ball

$$\text{prob} = \frac{\binom{n-1}{1}\binom{n}{2}(n-2)!}{(n-1)^n}$$

3. $E(|X|) = E(X) = \sum_{0}^{\infty} x P(X=x)$ → X is non-negative

$$= P(X=1) + 2P(X=2) + \cdots$$

$$= (P(X=1) + P(X=2) + \cdots)$$
$$+$$
$$(P(X)=2 + P(X=3) + \cdots)$$
$$+$$
$$\vdots$$

$$= P(X \geq 1) + P(X \geq 2) + \cdots$$

$$= \sum_{1}^{\infty} P(X \geq x)$$

4. $X \sim Geo(p)$

$$P(X=x) = \begin{cases} (p)(1-p)^{x-1} & , x = 1,2,\dots \\ 0 & , o.w \end{cases}$$

$$Y = \max(X, M)$$

① $\{Y = M\} = \{X \leq M\}$

② $\{Y = y\} = \{X = y\} \quad \forall\ y > M$

$$P(X \leq M) = \sum_{x=1}^{M} p(1-p)^{x-1}$$

$$= p[1 + (1-p) + \dots (1-p)^{M-1}]$$

$$= p \frac{[1-(1-p)^M]}{1-(1-p)}$$

$$= 1-(1-p)^M$$

$$P(Y=y) = \left\{ \begin{array}{ll} 1-(1-p)^M, & y = M \\ p(1-p)^{y-1} & y > M \\ 0 & \text{otherwise} \end{array} \right\}$$

5. $E(X) = \sum_{k=1}^{\infty} P(X \geq k) \geq \sum P(Y > k) = E(Y)$

6. 3 fair dice → independent

$$E(X) = \frac{1+2+3+4+5+6}{6}$$

$$= 1 + 1 + \frac{9}{6}$$

$$= 2 + 3/2 = 2 + 1.5 = 3.5$$

$$E(X+X+X) = 3E(X)$$

7. $E(XY)$ exist

$$E(XY) = \sum\sum xy\, P(X=x, Y=y)$$

$$= \sum x\, P(X=x) \left[\sum y P(Y=y \mid X=x)\right]$$

$$= \sum x P(X=x) \left[\sum P(Y \geq k \mid X=x)\right]$$

$$= \sum\sum x\, P(X=x, Y \geq k)$$

$$= \sum P(Y \geq k) \left[\sum x P(X=x \mid Y \geq k)\right]$$

$$= \sum P(Y \geq k) \left[\sum P(X \geq m \mid Y \geq k)\right]$$

$$= \sum\sum P(X \geq m, Y \geq k)$$

$$= E(XY)$$

8. let X be r.v with following property $P(X=n) = \begin{cases} \frac{c}{n^{k+2}} & ; n \in \mathbb{N} \\ 0 & ; \text{oth} \end{cases}$

c is some s.t $P(X=n)$ is a valid P.m.f

$$E|X|^k = c \sum_{n \geq 1} \frac{1}{n^2} < \infty$$

$$E|X|^{k+1} = c \sum_{n \geq 1} \frac{1}{n}$$ this diverges

9. n distinct balls, randomly into r distinct boxes.

$i = 1, 2, \ldots, n$

$$X_i = \begin{cases} 1 & \text{i-th box empty} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_i) = P(X_i = 1) = \frac{(r-1)^n}{r^n}$$

$$E(X_i^2) = P(X_i^2 = 1) = P(X_i = 1) = \frac{(r-1)^n}{r^n}$$

$$E(X_i, X_j) = \frac{(r-2)^n}{r^n} \ \forall i \neq j$$

N = total no of empty box.

$N = \sum_{i=1}^{r} X_i$ as X_i's are not independent

N ~ Not Bernoli

$$\text{var}(N) = E[(\sum X_i)^2] - E(\sum X_i)^2$$

$$E(N^2) = E\left[\sum X_i^2 - \sum\sum X_i X_j\right] - E[\sum X_i]^2$$

$$= r\frac{(r-1)^n}{r^n} + (r)(r-1)\frac{(r-2)^n}{r^n} - \left(\frac{r(r-1)^n}{r^n}\right)^2$$

10. $P[|X-Y| \leq M] = 1$

$$E(|Z|) = \sum^{\infty} |z| P(Z=z)$$

$$= \sum_{-M}^{M} |z| P(Z=z) + \sum |z| P(Z=z)$$

$$\leq M \sum_{-M}^{M} P(Z=z) + 0$$

$$\leq M < \infty$$

$$E|X| = E|X-Y+Y| \leq E|X-Y| + E|Y|$$

$$|E(X-Y)| \leq E|X-Y| \leq M$$

11. mean = 500

$P(X \geq 1000) \leq \frac{E(X)}{1000}$

$= 1/2$

$P(400 \leq X \leq 600) = P(|X-500| \leq 100) \geq 1 - \frac{Var(X)}{100^2} = 0.99$

12. $\mu_X = 100$

$\sigma^2 = 400$

$P(X \geq 120) \leq \frac{1}{2}$

$P(X - 100 \geq 20) \leq \frac{Var(X)}{20^2 + Var(X)}$

2. n balls
n boxes

(a) P that 1 box is empty

o o o . . . o

total n^n options

$\binom{n}{1} \times \binom{n-1}{1} \times \binom{n}{2} \times (n-2)!$ ← permutation of group of balls

choosing 2 balls

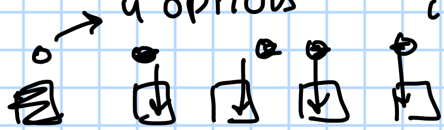
choosing 1 box from n-1

choosing 1 box

$P = \frac{\binom{n}{1}\binom{n-1}{1}\binom{n}{2}(n-2)!}{n^n}$

$\frac{\binom{n}{1}\binom{n-1}{1}\binom{n-1}{1}!}{(n-1)^n}$

(b) given 1 box empty



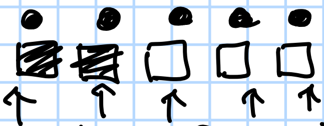
P n-1 boxes are not empty $= \frac{\binom{n-1}{1}\binom{n}{2}(n-2)!}{(n-1)^n}$

n-1 boxes to choose from

(c) given 1 box is empty

P(Box No 2 = empty | Box 1 is empty)

$= \frac{P(\text{Box 2 empty} \cap \text{Box 1 empty})}{P(\text{Box 1 empty})}$



(n-2) boxes
n balls
or $(n-2)^n$ total possibilities

$= \frac{\frac{1}{(n-2)^n}\left[\cancel{\binom{n}{2}}\binom{2}{1}(3)(3)\right]\cancel{(n-2)!}}{\binom{n-1}{1}\cancel{\binom{n}{2}}\cancel{(n-2)!} \times \frac{1}{(n-1)^n}}$

$= \left(\frac{n-1}{n-2}\right)^n \times \frac{9 \times 2}{(n-1)!} \times (n-2)!$

3. $E(X) = \sum_{i=0}^{\infty} x_i P(X = x_i) < \infty$

$$= \sum_{i=0}^{\infty} i \; P(X = i)$$

$$= P(X=1) + 2P(X=2) + 3P(X=3) + \cdots$$

$$= P(X=1) + P(X=2) + \cdots$$
$$+ P(X=2) + \cdots$$
$$+ \cdots$$
$$+ \cdots$$

$$= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + \cdots$$

$$= \sum_{k=1}^{\infty} P(X \geq k)$$

4. $X \sim Geo(p)$

$$P(X = x) = \begin{cases} (p)(1-p)^{x-1}, & x = 1, 2, \ldots \\ 0, & o.w \end{cases}$$

$Y = \max(X, M)$

then $\{X \leq M\} = \{Y = M\}$

$\{X = y\} = \{Y = y\} \quad \forall \; y > M$

$\{Y = y\} = \phi$ for $y < M$

$$\therefore P(Y = y) = \begin{cases} P(\{X = y\}) & ; y > M \\ P(\{X \leq M\}) & ; y = M \\ P(\phi) & ; y < M \end{cases}$$

$$= \begin{cases} (p)(1-p)^{y-1} & ; y > M \\ 1 - (1-p)^{M} & ; y = M \\ 0 & ; y < M \end{cases}$$

5. X is stochastically larger than Y if

$X \geq_{st} Y$ if $\forall\ t \in \mathbb{R}$

$P(X > t) \geq P(Y > t)$

<u>To prove</u> : $E(X) \geq E(Y)$

given $X \geq 0, Y \geq 0$

<u>proof</u>:

now as $P(X > t) \geq P(Y > t)\ \forall\ t \in \mathbb{R}$

$\Rightarrow \sum P(X \geq k) \geq \sum P(Y \geq t)\ \forall\ t \in \mathbb{N}$

$\Rightarrow E(X) \geq E(Y)$

$$E(X) = \sum_{k=1}^{\infty} P(X \geq k) \geq \sum P(Y > t) = E(Y)$$

6. X = no. after fair dice rolled

as 3 fair dice are rolled and independent

$$E(\underbrace{X + Y + Z}_{\text{fair dice rolled}}) = E(X) + E(Y) + E(Z) = 3E(X)$$

$$E(X) = \frac{1+2+3+4+5+6}{6}$$

$$= 3.5$$

$$3 \times 3.5 = 10.5$$

$$3E(X) = 10.5$$

7. $X \geq 0, Y \geq 0$ (int valued)

given: $E(XY)$ exist

<u>to prove</u>: $E(XY) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P(X \geq n, Y \geq m)$ $\left(\because E(XY) \text{ exist}\right)$

<u>proof</u>: $E(XY) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} nm\ P(X = n, Y = m)$

$$= \sum_{m=1}^{\infty} m \left(\sum_{n=1}^{\infty} n P(X = n, Y = m)\right)$$

$$= \sum_{m=1}^{\infty} m\ P(Y = m) \left[\sum_{n=1}^{\infty} n\ P(X = n \mid Y = m)\right]$$

$$= \sum_{m=1}^{\infty} m\ P(Y = m) \left[\sum_{n=1}^{\infty} P(X \geq n \mid Y = m)\right]$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m\, P(X \geq n, Y = m)$$

$$= \sum_{n=1}^{\infty} P(X \geq n) \left(\sum_{m=1}^{\infty} m\, P\left(Y = m \,\middle|\, X \geq n\right) \right)$$

$$= \sum_{n=1}^{\infty} P(X \geq n) \left(\sum_{m=1}^{\infty} P\left(Y \geq m \,\middle|\, X \geq n\right) \right)$$

$$E(XY) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P(X \geq n, Y \geq m)$$

8. X s.t k-th order $(k \geq 1, k \in \mathbb{N})$ exist but $(k+1)^{th}$ doesn't

$\uparrow$ discrete random variable

$E(X^k)$ exist or $\sum |x|^k p(x) < \infty$

$E(X^{k+1})$ does not exist or $\sum |x|^{k+1} p(x) \to \infty$

now let $P(X = n) = \begin{cases} \frac{c}{n^{k+2}} \; ; \text{ for } n \in \mathbb{N} \\ 0 \; ; \text{ otherwise} \end{cases}$

$$E(|X|^k) = \sum (n)^k \frac{c}{n^{k+2}} = \sum \frac{c}{n^2}$$

$\uparrow$ converges $(p > 1)$

$$E(|X|^{k+1}) = \sum (n)^{k+1} \frac{c}{n^{k+2}} = \sum \frac{c}{n} \text{ (diverges)}$$

$(p \leq 1)$

9. n distinct balls r boxes (distinct)

$X_i = \begin{cases} 1 & \text{if } i\text{th box is empty} \\ 0 & \text{otherwise} \end{cases}$

(a) $E(X_i) = \sum (1) P(X_i = 1)$

$= P(X_i = 1)$ ith box is empty

O O O O $\quad$ n options, $n-1$ options

$\uparrow$ empty

$$P(X_i = 1) = \frac{(n-1)^r}{(n)}$$

(b) $E(X_i X_j) = \sum (i)(j)\, P(X_i = i, X_j = j)$

$$= P(X_i = 1, X_j = 1)$$

or ith and jth boxes are empty

$$P(X_i = 1, X_j = 1) = E(X_i X_j) = \frac{(n-2)^r}{(n)^r}$$

(c) N = No. of empty boxes $= \sum_{i=1}^{n} X_i$

To find: $\text{var}\left(\sum_{i=1}^{n} X_i\right) = E\left[\left(\sum X_i\right)^2\right] - E\left(\sum X_i\right)^2$

now $E(X_i) = \frac{(n-1)^r}{n^r}$

$$E(\sum X_i) = \frac{n(n-1)^r}{n^r} = \frac{(n-1)^r}{n^{r-1}} = E(\sum X_i)$$

$$\frac{(n-1)^{2r}}{n^{2r-2}} = \left(E(\sum X_i)\right)^2$$

now, $E\left(\left(\sum X_i\right)^2\right) = E\left(\sum X_i^2 + 2\sum X_i X_j\right)$

$$= E(\sum X_i^2) + 2E(\sum X_i X_j)$$

$$= E(\sum X_i) + 2E(\sum X_i X_j)$$

$$E(\sum X_i) = \frac{n}{n^r}(n-1)^r$$

$$E(\sum X_i X_j) = (n)(n-1)\frac{(n-2)^r}{n^r}$$

or $\text{var}(N) = \frac{(n-1)^r}{n^{r-1}} + \frac{(n-1)(n-2)^r}{n^{r-1}} - \frac{(n-1)^{2r}}{n^{2r-2}}$

18. N no. of people entering

$$E(N) = 50$$

X_i: Amount of money spend by i^{th} customer

$E\left(\sum_{i=1}^{N} X_i\right)$ ← N is random

using $E(X) = E(E(X|Y))$

$$E\left(\sum_{i=1}^{N} X_i\right) = E\left(E\left(\sum_{i=1}^{N} X_i \,\middle|\, N=n\right)\right)$$

$$= E\left(E\left(\sum X_i\right)\right)$$

$$= E(10N)$$

$$= 10 \times 50 = 500$$

17. 3 coins — Head

Head

fair coins

2 coin tossed

$E(X|Y)$ → no of fair coins tossed

→ no of heads

Y can be 1 or 2

X can be 0, 1, 2

case I : Y takes 2

$\{HH, HT, TH, TT\}$ $P(X=0|Y=2) = 1/4$, $P(X=1, Y=2) = 1/2$, $P(X=2, Y=2) = 1/4$

case II: Y takes 1

$\{HH, HT\}$ $P(X=0|Y=1) = 0$

$P(X=1|Y=1) = 1/2$

$P(X=2|Y=1) = 1/2$

$$E(X|Y=y) = \begin{cases} 0 \times 0 + \frac{1}{2} \times 1 + \frac{1}{2} \times 2 & ; Y=1 \\ \frac{1}{4} \times 0 + \frac{1}{2} \times 1 + \frac{1}{4} \times 2 & ; Y=2 \end{cases}$$

$$= \sum x P(X=x|Y=y)$$

16. 4 balls

1 2 3 4

draw 2

if atleast one of the drawn balls has vale > 2

X can be 10 or -10

if Y_1 = first no drawn

Y_2 = second no drawn

$P(X=10|Y=1)$

has cases like: next ball 3 or 4

or $P(X=10|Y=1) = 2/3$

$P(X=10|Y=2) = 2/3$ → 1/3 $P(X=-10|Y=1) = 1/3$

$P(X=10|Y=3) = 1$ → 0

$P(X=10|Y=4) = 1$ → 0

$$E(X|Y) = \sum x P(X=x | Y=y)$$

$$= \begin{cases} 10(2/3) - 10(1/3) ; & Y=1 \\ 10(2/3) - 10(1/3) ; & Y=2 \\ 10(1) - 10(0) ; & Y=3 \\ 10(1) - 10(0) ; & Y=4 \end{cases}$$

$$= \begin{cases} 10/3 ; & Y=1 \\ 10/3 ; & Y=2 \\ 10 ; & Y=3 \\ 10 ; & Y=4 \end{cases}$$

15. $X, Y, Z \rightarrow$ discrete random variables on $(\Omega, \mathcal{F}, P)$

(a) $E(aY+bZ|X) = aE(Y/X) + bE(Z|X) \quad a,b \in \mathbb{R}$

as $E(aY+bZ|X) = \sum_y \sum_z (ay+bz) P(Y=y, Z=z | X=x)$

$$= a\left(\sum y P(Y=y | X=x)\right) + b\left(\sum z P(Z=z | X=x)\right)$$

$$= a E(Y/X) + b E(Z|X)$$

(b) $E(Y|X) = \sum y P(Y=y | X=x)$

as $y \geq 0$

and $P(Y=y | X=x) \geq 0$

$\Rightarrow E(Y|X) \geq 0$

14. $X \in \{2, 3, \ldots 12\}$

$Y \in \{1, \ldots 6\}$

Y_2: outcome on second

$$P(Y+Y_2 = x | Y=y)$$

$$= P(Y_2 = x-y | Y=y)$$

$$= P(Y_2 = x-y)$$

$$= \begin{cases} 1/6 ; & x>y, \; x-y \leq 6 \\ 0 ; & \text{otherwise} \end{cases}$$

$$E(X \mid Y=y) = \sum_{x=2}^{12} x\, P(X=x \mid Y=y)$$

$$= \sum_{x=y+1}^{y+6} x\left(\frac{1}{6}\right) + 0$$

$$= y + 7/2$$

13. $X+Y=m$

independent

$X, Y \to$ Binomial random variable

$$X, Y \overset{ind}{\sim} Bin(n, p)$$

$$X+Y \sim Bin(2n, p)$$

as $P(Z=z) = P(X+Y=z)$

$$= \underset{\text{by independence}}{\sum P(X=x,\ Y=z-x)}$$

$$P(X=x \mid X+Y=m) = \frac{P(X=x)\,P(Y=m-x)}{P(X+Y=m)}$$

$$= \frac{\binom{n}{x} p^x q^{n-x} \binom{n}{m-x} p^{m-x} q^{n-m+x}}{\binom{2n}{m} p^m q^{2n-m}}$$

$$= \frac{\binom{n}{x}\binom{n}{m-x}}{\binom{2n}{m}}$$

1) $0 \le x \le n$
2) $0 \le m-x \le n$

$$\max\{0, m-n\} \le x \le \min\{n, m\}$$

hypergeometric dist

$$E(X \mid X+Y=m) = \sum_{x=\max\{0,m-n\}}^{\min\{m,n\}} x \cdot \binom{n}{x}\binom{n}{m-x} \Big/ \binom{2n}{m}$$

$$= \frac{\sum_{x=1}^{m} \frac{n!}{(x-1)!\,(n-x)!}\binom{n}{m-x}}{\binom{2n}{m}}$$

$$= n \sum_{x=1}^{n} \frac{\frac{(n-1)!}{(x-1)!\,((n-1)-(x-1))!} \binom{n}{m-x}}{\binom{2n}{m}}$$

$$= m/2$$

10. $P(|X-Y| \leq M) = 1$

$$E|X-Y| \leq \sum |x-y| \, P(|X-Y| = r)$$

$$= M \sum P(|X-Y| = r)$$

$$= M$$

$$E(|X-Y|) \leq M$$

now

$$E|X| \leq \underset{<\infty}{E|X-Y|} + \underset{<\infty}{E|Y|}$$

also

$$|E(X)| - |E(Y)| \leq |E(X-Y)| \leq E(|X-Y|) \leq M$$

11. N = no of items produced in a week

$$E(N) = 500$$

(a) $P(N \geq 1000) \leq \frac{E(N)}{1000} = \frac{1}{2}$

(b) $\sigma^2 = 100$

$$P(400 \leq N \leq 600) = P(|N-500| \leq 100)$$

now $P(|N-500| > 100)$

$$\leq \frac{\sigma^2}{(100)^2}$$

$$P(|N-500| \leq 100)$$

$$= 1 - P(|N-500| > 100)$$

$$\geq 1 - \frac{\sigma^2}{(100)^2}$$

$$= 1 - \frac{\cancel{100}}{100 \cdot \cancel{100}}$$

$$= 0.99$$

$$P(|N-500| \leq 100) \geq 0.99$$

12. N = no of items in a factory

$E(N) = 100$

$\sigma^2 = 400$

To show: $P(X \geq 120) \leq 1/2$

proof: $P(X - 100 \geq 20) \leq \dfrac{\sigma^2}{a^2 + \sigma^2} = \dfrac{400}{400 + 400} = \dfrac{1}{2}$

13. $X, Y \rightarrow$ independent binomial random variables

$X, Y \sim Bin(n, p)$

now $X + Y \sim$ what?

$X \sim Bin(n, p)$
$Y \sim Bin(n, p)$

then $Z = X + Y$
is s.t

$$P(Z = z) = P(X + Y = z)$$

$$= \sum_{r=0}^{z} P(X = r, Y = z - r)$$

$$= \sum_{r=0}^{z} P(X = r)\, P(Y = z - r)$$

$$= \sum_{r=0}^{z} \binom{n}{r} (p)^{\cancel{r}} (1-p)^{n - \cancel{r}} \binom{n}{z-r} (p)^{z - \cancel{r}} (1-p)^{n - z + \cancel{r}}$$

$$= \sum_{r=0}^{z} \binom{n}{r} \binom{n}{z-r} (p)^{z} (1-p)^{2n-z}$$

$^nC_r \; ^nC_{z-r}$

$$(1+x)^{2n} = \cdots + \binom{2n}{r} x^r + \cdots$$

$$(1+x)^n (1+x)^n = \left(\cdots + \binom{n}{r} x^r + \cdots\right)$$

$$\left(\cdots + \binom{n}{r} x^{n-r} + \cdots\right)$$

x^z factor is : $\sum \binom{n}{r} \binom{n}{z-r} = \binom{2n}{z}$

so $\sum \binom{n}{r}\binom{n}{z-r} = \binom{2n}{z}$

or $X+Y \sim \text{Bin}(2n, p)$

now,

$$P(X=x \mid X+Y=m)$$

$$= \frac{P(X=x,\ Y=m-x)}{P(X+Y=m)}$$

$$= \frac{\binom{n}{x}\cancel{(p)^{x}}\cancel{(1-p)^{n-x}}\binom{n}{m-x}\cancel{(p)^{m-x}}\cancel{(1-p)^{n-m+x}}}{\binom{2n}{m}\cancel{(p)^{m}}\cancel{(1-p)^{2n-m}}}$$

$$= \frac{\binom{n}{x}\binom{n}{m-x}}{\binom{2n}{m}}$$

now $E(X \mid X+Y=m) = \sum x\, P(X=x \mid X+Y=m)$

$$= \sum_{x=0}^{m} \frac{\binom{n}{x}\binom{n}{m-x}}{\binom{2n}{m}}(x)$$

$$= \frac{1}{\binom{2n}{m}} \times \sum_{x=1}^{m} \binom{n}{x}\binom{n}{m-x} x$$

$$= \frac{1}{\binom{2n}{m}}\left[\frac{x \times (n)!}{(n-x)!\,(x)!} \times \frac{(n)!}{(m-x)!\,(n-m+x)!}\right]$$

$$= \frac{\sum x \binom{n}{x}\binom{n}{m-x}}{\sum \binom{n}{x}\binom{n}{m-x}}$$

$$= \frac{\sum x \frac{(n)!}{(n-x)!\,(x)!} \times \frac{(n)!}{(m-x)!\,(n-m+x)!}}{\binom{2n}{m}}$$

$$= n \sum \frac{x\,(n-1)!}{(n-x)!\,(x-1)!} \times \binom{n}{m-x} \Big/ \ldots$$

$$= n \sum \binom{n-1}{x-1} \binom{n}{m-x} \Big/ \ldots$$

now $\sum \binom{n-1}{x-1}\binom{n}{m-x}$ is

coeff of x^{m-1} in

$(1+x)^{n-1} \times (1+x)^{n}$

or $= \binom{2n-1}{m-1}$

so $E(X \mid X+Y=m) = \dfrac{n \times \binom{2n-1}{m-1}}{\binom{2n}{m}}$

$$= \frac{n \times \frac{(2n-1)!}{(m-1)!\,\cancel{(2n-m)!}}}{\frac{(2n)!}{\cancel{(2n-m)!}\,(m)!}}$$

$$= \frac{\cancel{n}}{(2\cancel{n})} \times (m) = \frac{m}{2}$$

14. fair dice twice roll

X = sum of outcomes and Y = outcome of first roll

$$E(X \mid Y=y) = \sum_{x=2}^{12} x \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$\frac{P(X=x, Y=y)}{P(Y=y)} = \frac{P(X=x, Y=y)}{P(Y=y)}$$

1 to 6

or $P(X=x, Y=y) = P(Y_2 = x-y, Y=y)$

$$= \frac{1}{6}\cdot\frac{1}{6}$$

$$\frac{P(X=x, Y=y)}{P(Y=y)} = \frac{\frac{1}{6}\cdot\cancel{\frac{1}{6}}}{\cancel{\frac{1}{6}}}$$

$$= \frac{1}{6}$$

for $y \in \{1, 2, \dots 6\}$ $\quad x > y$, $x - y \leq 6$

or $E(X \mid Y=y) = \sum_{x=2}^{12} (x)\left(\frac{1}{6}\right)$

$$= \sum_{x=y+1}^{x=y+6} \left(\frac{x}{6}\right)$$

$$= \frac{1}{6}\left(y+1 + y+2 + y+3 + y+4 + y+5 + y+6\right)$$

$$= \frac{1}{6}\left(6y + \frac{(6)(7)}{2}\right)$$

$$= y + 7/2$$

15. $X, Y, Z \quad (\Omega, \mathcal{F}, P)$

(a) $E(aY + bZ \mid X) = aE(Y \mid X) + bE(Z \mid X) \quad a, b \in \mathbb{R}$

now

$$E(aY + bZ \mid X) = \sum\sum (ay + bz)\, P(Y=y, Z=z \mid X=x)$$

$$= \sum ay\, P(Y=y \mid X=x) + \sum bz\, P(Z=z \mid X=x)$$

$$E(aY + bZ \mid X) = aE(Y/X) + bE(Z/X)$$

(b) $E(Y|X) \geq 0$ if $Y \geq 0$

as

$$E(Y|X) = \sum y \, P(Y=y \mid X=x)$$

as $Y \geq 0$

$\Rightarrow y \geq 0$

so $\sum y \, P(Y=y \mid X=x) \geq 0$

as P is always ≥ 0

(c) $E(1|X) = 1$

proof : $E(1|X) = \sum (1) \, P(1/X=x)$

$= P(1=1/X=x)$

as independent

now $\dfrac{P(1=1 \text{ and } X=x)}{P(X=x)}$

$= \dfrac{\cancel{P(X=x)}}{\cancel{P(X=x)}} = 1$

so $E(1/X) = 1$

16. four balls 1, 2, 3, 4

2 balls drawn

X = Price won

Y = ball drawn

$P(X=10 \mid Y=1) = 2/3 \leftrightarrow p' = 1/3$

$P(X=10 \mid Y=2) = 2/3 \leftrightarrow p' = 1/3$

$P(X=10 \mid Y=3) = 1 \leftrightarrow p' = 0$

$P(X=10 \mid Y=4) = 1 \leftrightarrow p' = 0$

now $E(X|Y) = \begin{cases} 10/3 & ; \quad y = 1 \text{ or } 2 \\ 10 & ; \quad y = 3 \text{ or } 4 \end{cases}$

17. → 3 coins

HH   HT   HT

X = no of heads = 0, 1, or 2

Y = no of fair coins = 1, or 2

$Y = 1$ then $\{HH, HT\}$

$Y = 2$ then $\{HH, HT, TH, TT$

now $E(X|Y) = \begin{cases} 0 + 1\left(\frac{1}{2}\right) + \frac{1}{2} & ; \; y = 1 \\ 0 + 1 \cdot \frac{1}{2} + 2\,\frac{1}{4} & ; \; y = 2 \end{cases}$

$= \begin{cases} 1 & ; \; y = 1 \\ 1 & ; \; y = 2 \end{cases}$

$E(X|Y) = 1$

18. N = no. of people entering

$E(N) = 50$

$\$$ = Amount of money spend
(not random var)

$E(\$) = 10$

$\$$ = Amout of money spend

$\sum_{r=1}^{N} \$$ then a new random variable

fix N to get

$E(\underset{\text{total}}{\$} \,|\, N = n) = E(\underbrace{\$ + \$ + \cdots \$}_{n \text{ times}})$

$= 10n$

or $E(\$_{total} \mid N) = 10N$

now

$$E(E(\$_{total} \mid N)) = E(\$_{total})$$

$$= E(10N)$$

$$= 10 \times 50$$

$$= 500$$

$$E(\$_{total}) = 500$$

Tutorial-5:

$X \to$ random variable if F_X is cont function

Defn: X r.v is ab cont with density $f_X : \mathbb{R} \to [0,\infty)$

st

$$P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt, \ \forall x \in \mathbb{R}$$

note: $P(X = y) = 0 \ \forall y \in \mathbb{R}$ or X is cont r.v

Normal dist: $X \sim N(\mu, \sigma^2)$

when $\mu, \sigma \in \mathbb{R}$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx_{\infty}$

Note: $\displaystyle\int_{-\infty} e^{-x^2} = \sqrt{\pi}$

$$u = \left(\frac{x-\mu}{\sqrt{2\sigma^2}}\right) \qquad du = \frac{dx}{\sqrt{2\sigma^2}}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \times \sqrt{2\sigma^2} \times e^{-u^2} du = 1$$

(ii) $X \sim N(0,1)$ then

$$P(X \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi(1)}} e^{\frac{-(x)^2}{2}} dx$$

To show: $a + bX$ has $N(a, b^2)$

now $a + bX \to$ R.V

$$P(a + bX \le x)$$

$$= P\left(X \le \frac{x-a}{b}\right)$$

$$= \int_{-\infty}^{\frac{x-a}{b}} \frac{1}{\sqrt{2\pi(1)}} e^{-\frac{t^2}{2}} dt$$

here

$G_Y(y) = P\left(X \le \frac{y-a}{b}\right)$

$G_Y(y) = F_X\left(\frac{y-a}{b}\right)$

$g_Y(y) = \frac{d}{dy} G_Y(y)$

$= \frac{d}{dy} F_X\left(\frac{y-a}{b}\right)$

$= f_X\left(\frac{y-b}{a}\right) \frac{d}{dy}\left(\frac{y-a}{b}\right)$

if replace with u inside, insted of t

or $u = bt + a$

$-\infty$ to x

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{u-a}{b}\right)^2 \times \frac{1}{2}} \frac{1}{b} du$$

Pdf

$g_Y(y) = f_X\left(\frac{y-b}{a}\right)\left(\frac{1}{b}\right)$

$$P(a+bX \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{(u-a)^2}{2b^2}} du$$

$$\int_{-\infty}^{y} f_x\left(\frac{y-b}{a}\right)\left(\frac{1}{b}\right)dy$$

or $a+bX \sim N(a, b^2)$

(iii) $X \sim N(0,1)$

$$Y = e^X$$

density of $Y = e^X$

X density $f_X(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$P(X \le x) = \int_{-\infty}^{x} f_X(t)dt$$

$$P(e^X \le x) = P(X \le \log x)$$

$$= \int_{-\infty}^{\log x} f_X(t)dt$$

$$= \int_{0}^{x} f_X(\log u)\, u\, du$$

$u = e^t$

then $u = e^{-\infty} = 0$

$du = e^t dt$

$$= \int_0^x \underbrace{f_X(\log u)\frac{1}{u}}_{\text{density of } e^x} du$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{(\log u)^2}{2}} \times \frac{1}{u}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\log u)^2} \times \frac{1}{u}$$

or $f_Y(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-1\cdot(\log x - \mu)^2 \cdot \frac{1}{2\sigma^2}}$

$G_Y(y) = P(e^X \le y)$

$= P(X \le \ln y)$

$G_Y(y) = F_X(\ln y)$

$\frac{d}{dy} G_Y(y) = \frac{d}{dy} F_X(\ln y)$

$= g_Y(y)$

$g_Y(y) = f_X(\ln y)\left[\frac{1}{y}\right]$

$$= \begin{cases} \frac{1}{\sqrt{2\pi y^2}} e^{\left(\frac{-1}{2}(\ln y)^2\right)} dy & ; y>0 \\ 0 & ; \text{otherwise} \end{cases}$$

__Exponential__ __dist__ :

$T \sim Exp(\lambda)$

$T \sim Exp(\lambda)$
$0 \leq \lambda < \infty$

$$P(T > t) = e^{-\lambda t} \quad \text{for } t \geq 0$$

$\lambda > 0$ then

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P(T \leq t) = \int_{-\infty}^{t} f_T(t)\,dt = \int_0^t \lambda e^{-\lambda t}\,dt$$

$$= \cancel{\lambda}\left(\frac{e^{-\lambda t}}{-\cancel{\lambda}}\right)_0^t$$

$$P(T \leq t) = 1 - e^{-\lambda t}$$

$$P(T > t) = e^{-\lambda t}$$

1. $X \sim$ RV with

$$f(x) = \begin{cases} c(1-x^2) & ; -1 < x < 1 \\ 0 & ; \text{else} \end{cases}$$

(a) $$P(X \leq x) = \int_{-1}^{1} c(1-x^2)\,dx = \left| cx - \frac{x^3}{3}c \right|_{-1}^{1}$$

$$= c\left(1 - \frac{1}{3}\right) - c\left(-1 + \frac{1}{3}\right)$$

$$= c\left(1 - \frac{1}{3}\right) + c\left(1 - \frac{1}{3}\right)$$

$$= 2 \times \frac{2}{3} \times c = \frac{4}{3}c = 1$$

$$c = \frac{3}{4}$$

(b) X dist function

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$$

for $x \leq -1$

$$F_X(x) = 0$$

$$x \geq 1$$

$$F_X(x) = 1$$

and for $-1 < x < 1 \Rightarrow F_X(x) = \int_{-1}^{x} \left(\frac{3}{4}\right)(1-x^2)\,dx$

$$= \frac{3}{4}\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{x}$$

$$= \frac{3}{4}\left(x-\frac{x^3}{3}\right) - \frac{3}{4}\left(-1+\frac{1}{3}\right)$$

$$= \frac{3}{4}\left(x - \frac{x^3}{3} + \left(+\frac{2}{3}\right)\right)$$

$$= \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right)$$

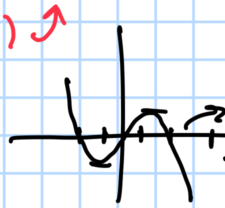
2. $f(x) = \begin{cases} c\,(2x - x^3) & ; \ 0 < x < 5/2 \\ 0 & ; \ \text{otherwise} \end{cases}$

for $0 < x < 5/2$

$2 - 3x^2 = 0$

$\Rightarrow x = \pm\sqrt{\frac{2}{3}} \rightarrow$ max/min

$F'(x) > 0$ inc, $F(x) \nearrow$ density or $f_X(x) > 0$



$\sqrt{2} = 1.313\ldots$

$\frac{5}{2} = 2.5$

$2x = x^3$ at $x = 0$, $x = \pm\sqrt{2}$

or as negative f is not a p.d.f

3. $f(x) = c e^{-x^2/2}$

$$\int_{-\infty}^{\infty} c e^{-x^2/2}\, dx = 1$$

or let $u = x/\sqrt{2}$

then $\sqrt{2}\, du = dx$

$$\int_{-\infty}^{\infty} \sqrt{2}\, c\, e^{-u^2}\, du = c\sqrt{2} \times \sqrt{\pi} = 1$$

$$c = \frac{1}{\sqrt{2\pi}}$$

4.

Train A	Train B
7 Am	7:05
7:15	7:20
7:30	7:35
7:45	7:50
8:00	8:05

→ uniform probability

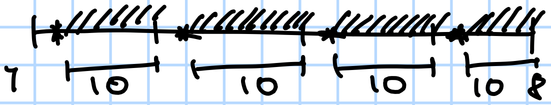
we have $f_X(x) = c$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{7}^{8} f_X(x)\,dx = C(1) = 1$$

$\underline{C=1}$

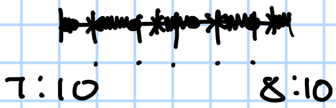
or if in min then $c = \frac{1}{60}$

$$\int_{7:00}^{8:00} \frac{1}{60}\,dx = \frac{1}{60} \times 60 = 1$$



the shaded time required

$$\frac{40}{60} = \frac{2}{3}$$



$$5 \quad 10 \quad 10 \quad 10 \quad 5 = \frac{40}{60} = \frac{2}{3}$$

5. $X \sim U(0,1)$

$$Y = -\lambda^{-1} \log(1-X)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$P(Y \le x) = P\left(-\lambda^{-1}\log(1-X) \le x\right)$$

$$= P\left(-\log(1-X) \le \lambda x\right)$$

$$= P\left(\log \frac{1}{1-X} \le \lambda x\right)$$

$$= P\left(\frac{1}{1-X} \le e^{\lambda x}\right)$$

$$= P\left(e^{-\lambda x} \le 1-X\right)$$

$$= P\left(e^{-\lambda x} - 1 \le -X\right)$$

$F(x) = x, \; 0 < x < 1$

$$P[Y \le y] = P\left[-\frac{1}{\lambda}\ln(1-X) \le y\right]$$

$$= P\left[\ln(1-X) \ge -\lambda y\right]$$

$$= P\left(1-X \ge e^{-\lambda y}\right)$$

$$= P\left(X \le 1-e^{-\lambda y}\right)$$

$$= 1-e^{-\lambda y}, \; y > 0$$

$$= P\left(X \leq -e^{-\lambda x}+1\right)$$

$$= P\left(X \leq 1-e^{-\lambda x}\right)$$

$$= \int_{-\infty}^{1-e^{-\lambda x}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

now $1-e^{-\lambda t} = u$

$1-u = e^{-\lambda t}$

$-\lambda t = \log(1-u)$

$t = -\lambda^{-1}\log(1-u)$

$dt = +\lambda^{-1}\left(\frac{1}{1-u}\right)(+du)$

$1-e^{-(-\infty)} \to -\infty$

$dt = \lambda^{-1}\left(\frac{1}{1-u}\right) du$

$$\underbrace{\int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{(-\lambda^{-1}\log(1-u))^2}{2}} \lambda^{-1}\left(\frac{1}{1-u}\right) du}$$

$$f_Y(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\lambda^{-2})(\log(1-u))^2} \times \frac{1}{1-u} \times \frac{1}{\lambda}$$

6. X cont rad variable f density

$$P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

$$P(X \geq x) = 1 - \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{\infty} f(t)\,dt - \int_{-\infty}^{x} f(t)\,dt$$

$$P(X \geq x) = \int_{x}^{\infty} f(t)\,dt$$

$P(Y \leq y)$

$= P(|X| \leq y)$

$= F_X(y) - F_X(-y)$

then diff

now $P(|X| \le x) = P(-x \le X \le x)$

$$= \int_{-x}^{x} f(x)\,dx = \int_0^x 2f(x)\,dx$$

$$f_Y(y) = \begin{cases} 0 & ; x < 0 \\ 2f & ; x > 0 \end{cases}$$

7. X be positive random variable with f

$$P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x f(t)\,dt$$

$$f(t) = \begin{cases} f & ; x > 0 \\ 0 & ; x \le 0 \end{cases}$$

$$P(Y \le x) = P\left(\frac{1}{1+X} \le x\right)$$

$$= P\left(\frac{1}{x} \le 1+X\right)$$

$$= P\left(\frac{1}{x} - 1 \le X\right)$$

$$= P\left(X \ge \frac{1}{x} - 1\right)$$

$$= 1 - P\left(X \le \frac{1}{x} - 1\right)$$

now $\frac{1}{x} > 1 \Rightarrow 1 > x$
if $x > 1$ then

0

$$= 1 - \int_0^{\frac{1}{x} - 1} f(t)\,dt$$

doubt

$\underline{X > 0}$ c.r.v

$f_X(x) = f(x)$

$Y = \frac{1}{1+X}$

$$P(Y \le y) = P\left(\frac{1}{1+X} \le y\right)$$

$$= P(1 \le y + yX)$$

$$= P(1 - y \le yX)$$

$$= P\left(X \ge \frac{1-y}{y}\right)$$

$$= \int_{\frac{1-y}{y}}^{\infty} f(x)\,dx = \int_{\frac{1}{y}-1}^{\infty} f(x)\,dx$$

$\frac{1}{1+x} = u$

$-\frac{1}{(1+x)^2}\,dx = du$

$dx = -\frac{du}{u^2}$

$$\int_0^y f\left(\frac{1}{u} - 1\right)\frac{du}{u^2} = P(Y \le y)$$

$$f_Y(y) = \frac{d}{dy} \int_0^y f\left(\frac{1}{u} - 1\right)\frac{du}{u^2}$$

$y \in (0,1) \longrightarrow f_Y(y) = f\left(\frac{1}{y} - 1\right)\frac{1}{y^2}$

8. $[0,a]$

X = dist from origin

$$P(X \le x) = \int_0^x \frac{1}{a}\,dx$$

$$\underbrace{f(x)}= \frac{1}{a}$$

$$P(X > a/2) = P(Y = a/2) = \int_{a/2}^{a} \frac{1}{a}\,dx = \frac{1}{2}$$

$Y = \min\{X, a/2\}$

$$P(Y \le y) = P\left(\min\{X, a/2\} \le y\right)$$

$$= 1 - P\left(\min\{X, a/2\} > y\right)$$

$$= 1 - P\left(X > y, a/2 > y\right)$$

$$f_Y(y) = \begin{cases} \frac{1}{a} & 0 \le y < a/2 \\ 1/2 & y = a/2 \\ 0 & \text{oth} \end{cases}$$

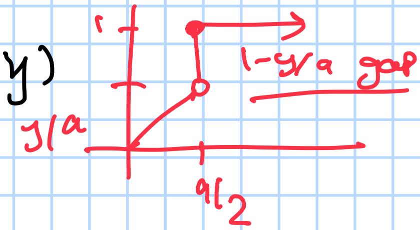
$$= 1 - P\left(X > y\right) \times \frac{1}{2}$$

$$= 1 - \frac{1}{2}\left(1 - \int_0^y \frac{1}{a}\,dx\right)$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y/a & 0 \le y < a/2 \\ 1 & y \ge a/2 \end{cases}$$

$$= 1 - \frac{1}{2} + \frac{1}{2}\int_0^y \frac{1}{y}\,dx$$

$$P(Y \le y) = \frac{1}{2} + \frac{1}{2}\int_0^y \frac{1}{y}\,dx = F_Y(y)$$



(mixed)

9. X $\quad F$ is strict inc

$$Y = F(X)$$

$$Y = F(X)$$

$$P(Y \le y) = P\left(F(X) \le y\right)$$

$$= P\left(X \le F^{-1}(y)\right)$$

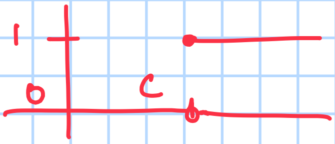
$$= F\left(F^{-1}(y)\right)$$

$$P(Y \le y) = y$$

$\therefore$ uniform dist

10. $X \sim$ disc random var

F



$$F(x) = \begin{cases} 0 ; & x < c \\ 1 ; & x \geq c \end{cases}$$

$F(X)$ is uniform dist on $(0,1)$ NO

doubt

now $Y = F(X)$

F is not invertible

$$P(Y \leq y) = P(F(X) \leq y)$$

11. $X \sim Exp(\lambda)$ $\lambda > 0$

Y ↑ int valued random var

$Y = \lfloor X \rfloor$

also $Y \geq 0$

$$P(Y \leq y) = \begin{cases} 0 ; & y < 0 \\ P(Y \leq \lfloor y \rfloor) ; & y > 0 \end{cases}$$

now

$$P(Y = \underset{m}{\underset{\shortparallel}{\lfloor y \rfloor}}) = P(\underbrace{m}_{\text{integer}} \leq X < m+1)$$

$$P(X > x) = e^{-\lambda t}$$

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} ; & t \geq 0 \\ 0 ; & \text{otherwise} \end{cases}$$

$$P(X \leq x) = 1 - e^{-\lambda x}$$

$$P(X > x) = e^{-\lambda x}$$

$$P(m \leq X < m+1) = \int_m^{m+1} (\lambda e^{-\lambda t}) dt$$

$$= \lambda \left(\frac{e^{-\lambda t}}{-\lambda} \right)_m^{m+1}$$

$$= -\left[e^{-\lambda(m+1)} \right] + e^{-\lambda(m)}$$

$$= e^{-\lambda m} \left[1 - e^{-\lambda} \right]$$

$$P(Y=m) = e^{-\lambda m}[1-e^{-\lambda}]$$

$$\sum_{m=0}^{\infty} P(Y=m) = \sum_{0}^{\infty} e^{-\lambda m}\left[1-\frac{1}{e^{\lambda}}\right]$$

$$= \left(1-\frac{1}{e^{\lambda}}\right)\left(1+e^{-\lambda}+e^{-2\lambda}+\cdots\right)$$

$$= \left(1-\frac{1}{e^{\lambda}}\right)\left(\frac{1}{1-e^{-\lambda}}\right)$$

$$= \cancel{\left(1-\frac{1}{e^{\lambda}}\right)}\left(\frac{1}{\cancel{1-\frac{1}{e^{\lambda}}}}\right)$$

$$= 1$$

$$\therefore P(Y=m) = e^{-\lambda m}\left[1-\frac{1}{e^{\lambda}}\right]$$

<u>cont function</u> :

$g: A \to \mathbb{R}$ some c cont.

if $\forall \varepsilon > 0, \exists \delta > 0$

$$f(B(x,\delta)) \subseteq B(f(x),\varepsilon)$$

<u>Note</u> : $(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_o X^{-1})$

$P_o X^{-1} \downarrow$ distn of X

$$B = (-\infty, x] \subseteq \mathcal{B}(\mathbb{R}) \quad \forall x \in \mathbb{R}$$

$$F_X(x) = P_o X^{-1}(B) = P(X \leq x)$$

(i) $\forall x < y$ in $\mathbb{R}$, $F_X(x) \leq F_X(y)$

(ii) $x \to -\infty$, $x \to \infty$
$\quad\;\; 0 \qquad\qquad 1$

(iii) $\lim_{h \to 0^+} F_X(x+h) = F_X(x)$

<u>Not</u> <u>left cont</u> :

$$D_n = \{\omega \in \Omega \mid X(\omega) \leq x - 1/n\}$$

$$\lim_{n\to\infty} D_n = \bigcup_{n\geq 1} D_n = \{\omega \in \Omega \mid X(\omega) < x\}$$

$$P(\lim_{n\to\infty} D_n) = \lim_{n\to\infty} P(D_n) \Rightarrow P(X < x) = \lim_{n\to\infty} F_X(x - \tfrac{1}{n})$$

$$\Rightarrow F_X(x) - P(X=x) = \lim_{h\to 0} F_X(x-h)$$

now if $P(X=x) = 0 \ \forall x$

then $F_X(x) = \lim_{h\to 0} F_X(x-h)$

$= \lim_{h\to 0} F_X(x+h)$

so X is c.r.v when CDF F_X is cont everywhere

$P(X=x) = 0 \ \forall x \in \mathbb{R}$

Task : suppose X is a discrete r.v :

$D_{F_X} = \{$set of Discont of $F_X(x)\}$

$= \{x \in \mathbb{R} \mid P(X=x) > 0\}$

To prove : D_{F_X} is almost countable

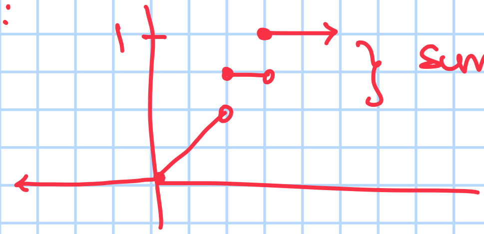
Remark : ① X is c.r.v is decided by cdf of $F_X(\cdot)$

② Discrete / cont are extremes

mixture of r.v :

sum of jumps $\neq 1$

eg :



$\}$ sum of jumps $= 1/2$

Note : singular cont dist. function (cont but no density)

cantor dist function

Absolutely cont fn :

$g : [a,b] \to \mathbb{R}$

(a_k, b_k) $k = 1(1)n$

$\to$ finite pairwise disjoint partitions

g is A.C on $[a,b]$ if $\forall \varepsilon > 0, \exists \delta > 0$

$$\sum_{1}^{n} (b_k - a_k) < \delta \Rightarrow \sum_{1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

first fundamental theorem of calculus :

if g is A.C on $[a,b]$ then $\exists$ a measurable f on $[a,b] \ni g(x) = g(a) + \int_a^x f(t)\,dt$

$\forall\ x \in [a,b]$

& $g'(x) = f(x)$

cdf of F_X to be AC

plug $a = -\infty$
$g = F_X(\ )$

$$F_X(x) = 0 + \int_{-\infty}^{x} f(t)\,dt$$

$$P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

Note: $X \sim N(\mu, \sigma^2)$

$$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \ ; x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0 \\ 0 \ ; \text{otherwise} \end{cases}$$

(a) $$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2}\,dt$$

$$= \sqrt{2/\pi} \int_0^{\infty} e^{-t^2/2}\,dt$$

$y = t^2/2$
$dy = \cancel{2}t/\cancel{2}\,dt$
$= t\,dt$

$$\Rightarrow \sqrt{2/\pi} \int_0^{\infty} \frac{e^{-y} y^{\frac{1}{2}-1}}{\sqrt{2}}\,dy$$

Note: $\Gamma n = (n-1)!$

$$= \int_0^{\infty} e^{-x} x^{n-1}\,dx$$

and $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

so $\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y} y^{-1/2} \, dy$

$= \frac{\cancel{\sqrt{\pi}}}{\cancel{\sqrt{\pi}}} = 1$

Tutorial-6 :

1. $f(x,y) = \begin{cases} \frac{6-x-y}{8} ; 0<x<2, \quad 2<y<4 \\ 0 \quad ; \text{else} \end{cases}$

$$P(X<1, Y<3) = \int_{-\infty}^{3} \left[\int_{-\infty}^{1} (f(x,y))\, dx \right] dy$$

$$= \int_{-\infty}^{3} \left[\int_{0}^{1} (f(x,y))\, dx \right] dy$$

$$= \int_{2}^{3} \int_{0}^{1} \left[\frac{6-x-y}{8} \right] dx\, dy$$

$$= \int_{2}^{3} \left[\frac{6 - \frac{1}{2} - y}{8} \right] dy$$

$$= \int_{2}^{3} \left[\frac{\frac{11}{2} - y}{8} \right] dy = \frac{11}{16} \times [1] - \frac{1}{16}[9-4]$$

$$= \frac{11}{16} - \frac{5}{16} = \frac{6}{16} = \frac{3}{8}$$

$P(X+Y \leq 3)$

$\to x+y \leq 3$

$0 < x < 2$

$2 < y < 4$

$0 < x \leq 3-y$

$2 < y < 3$

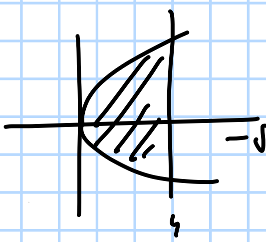
then

$$\int_{2}^{3} \int_{0}^{3-y} \left(\frac{6-x-y}{8} \right) dx\, dy = \int_{2}^{3} \left[\frac{6x - \frac{x^2}{2} - xy}{8} \right]_{0}^{3-y} dy$$

$$= \int_{2}^{3} \left[\frac{6(3-y) - \frac{1}{2}(3-y)^2 - (3-y)y}{8} \right] dy$$

$$= \int_{2}^{3} \left[\frac{18 - 6y - \frac{1}{2}(9+y^2-6y) - 3y + y^2}{8} \right] dy$$

2.



$0 \leq x \leq 4$

$-\sqrt{x} \leq y \leq \sqrt{x}$

uniformly dist for the area

or

$f(x,y) = c \leftarrow$ const

$$\int_0^4\int_{-\sqrt{x}}^{\sqrt{x}} c \, dy\,dx = \int_0^4 c[2\sqrt{x}]\,dx = 2c\left[\frac{x^{3/2}}{3/2}\right]_0^4 = \frac{4}{3}\times c \times (2)^3$$

$$= \frac{32}{3}c = 1$$

$$c = \frac{3}{32}$$

$$P(X<3, Y<0) = \int_0^3\int_{-\sqrt{x}}^{0} f(x,y)\,dy\,dx$$

3. $f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)$

$0 < x < 1$
$0 < y < 2$

(a)

$$\frac{6}{7}\int_0^1 \left(x^2 + \frac{xy}{2}\right)dx = \frac{6}{7}\left[\frac{1}{3} + \frac{1}{4}y\right]$$

$$= \frac{2}{7} + \frac{3}{14}y$$

$$\int_0^2 \frac{2}{7} + \frac{3}{14}(y)\,dy = \frac{2}{7}\times 2 + \frac{3}{\cancel{14}7}\times[\cancel{2}]$$

$$= 1$$

(b) $P(X>Y)$ $\quad y < x < 1$, $0 < y < 1$ $\to \int_0^1\int_y^1 f(x,y)\,dx\,dy$

Note: as $X > Y$

$$P(X>Y) = \int_0^1\int_0^x \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)dy\,dx$$

as $0 < x < 1$
$0 < y < 1$
but $x > y$
or $0 < y < x$
$0 < x < 1$

(c) $$P(X) = \int_0^x\int_0^2 f(x,y)\,dy\,dx$$

$$= \int_0^x \frac{6}{7}\left(x^2(2) + \frac{x}{\cancel{2}}\times\frac{1}{\cancel{2}}\times\cancel{4}\right)dx$$

$$= \int_0^x \frac{6}{7}\left(2x^2 + x\right)dx$$

<u>check</u>: $\int_0^1 \frac{6}{7}\left(2x^2 + x\right)dx = 1$

4. $f(x,y) = \begin{cases} x+y \; ; & 0<x<1, \; 0<y<1 \\ 0 \; ; & \text{else} \end{cases}$

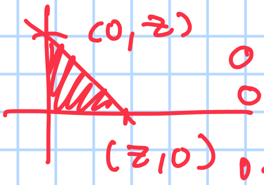
(a) $F_X(x) = \int_0^x\int_0^1 (x+y)\,dx$

(b) density of X+Y

$= P(X+Y \leq r)$

$P(X+Y \leq r)$ doubt

$z = x+y$



$f(x,y) = x+y$

$0 < x < 1$

$0 < y < 1$

$F_Z(z) = \begin{cases} 0 & z < 0 \\ P(Z+Y \leq Z) & z \in (0,1) \\ 1 & z \geq 2 \end{cases}$

$0 < z < 1$

$P(Z \leq z) = \int_0^{z} \int_0^{z-y} (x+y)\, dx\, dy$

$x+y \leq z$

$0 \leq x \leq z-y$

$0 \leq y \leq z$

$1 \leq z \leq 2$

$P(x+y \leq z)$

$= 1 - P(x+y > z)$

$= 1 - \int\int$

$x+y > z$

$1 > x > z-y$

$1 > y > z-1$

$P(X+Y \leq z)$

$= 1 - \int_{z-1} \int_{z-y}^{1} (x+y)\, dx\, dy$

5. $f(x,y) = ce^{-(x^2 - xy + 4y^2)/2}$

$$\int_{-\infty}^{\infty} ce^{-2y^2} \int_{-\infty}^{\infty} e^{(-x^2+xy)/2}\, dx\, dy$$

$$= \int_{-\infty}^{\infty} ce^{-2y^2} \int_{-\infty}^{\infty} e^{-\frac{\left(x^2 - \frac{2xy}{2} + \frac{y^2}{4}\right) + \frac{y^2}{4}}{2}}\, dx\, dy$$

const

$$= \int_{-\infty}^{\infty} ce^{-2y^2 + \frac{y^2}{8}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{y}{2}\right)^2 \times \frac{1}{2}}\, dx\, dy$$

$\frac{x}{\sqrt{2}} = t$

$dx = dt\sqrt{2}$

$$c\int_{-\infty}^{\infty} e^{-15y^2/8} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(x - y/2)^2}}{\sqrt{2\pi}}}_{1}\, dx\, dy = \frac{1}{\sqrt{2\pi}}$$

$$= \int_{-\infty}^{\infty} ce^{-2y^2 + y^2/8} \times \sqrt{2} \times \sqrt{\pi}\, dy$$

$$= c \times \sqrt{2} \times \sqrt{\pi} \int_{-\infty}^{\infty} e^{-15/8\, y^2}\, dy$$

$$\sqrt{2\pi}\, c \int_{-\infty}^{\infty} e^{-\frac{15}{8}y^2}\, dy = 1$$

$\sqrt{\frac{15}{8}}\, y = t$

$dy = \sqrt{\frac{8}{15}}\, dt$

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$$

$$\left(\sqrt{\frac{15}{8}}\, x\right) = u$$

$$= c \times \sqrt{2} \times \sqrt{\pi} \times \sqrt{\pi} \times \sqrt{\frac{8}{15}}$$

$= 1$

$c \times \frac{4}{\sqrt{15}} \times \pi = 1$

$c = \frac{\sqrt{15}}{4\pi}$

(b) $f_X(x) = \int_{-\infty}^{x} \left(\int_{-\infty}^{\infty} f(x,y)\, dy \right) dx$

$f_Y(y) = \int_{-\infty}^{y} \left(\int_{-\infty}^{\infty} f(x,y)\, dx \right) dy$

6. $f(x,y) = \begin{cases} 24xy ; & 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq x+y \leq 1 \\ 0 ; & \text{else} \end{cases}$

$0 \leq y \leq 1-x$

$0 \leq x \leq 1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = \int_0^1 \int_0^{1-x} 24xy\, dy\, dx$$

$$= \int_0^1 24x \times x \times \left(\frac{y^2}{2}\right)\Big|_0^{1-x} dx = 12\int_0^1 x\,[1-x]^2\, dx$$

$$= 12\int_0^1 x(1+x^2-2x)\,dx$$

$$= 12\left[\frac{x^2}{2}+\frac{x^4}{4}-\frac{2x^3}{3}\right]_0^1$$

$$= 12\left[\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right]$$

$$= 6+3-8 = 1$$

$\therefore$ Yes it is a joint density function

$$F_Y(y) = \int_0^y\int_0^{1-y} 24(xy)\,dx\,dy = \int_0^y \left[24\frac{x^2}{2}y\right]_0^{1-y} dy$$

$0 \leq y \leq 1$

$$= \int_0^y [12x^2y]_0^{1-y}\,dy$$

$$= \int_0^y [12][1+y^2-2y]\,y\,dy$$

$$= \int_0^y \underbrace{12(y+y^3-2y^2)}_{\text{marginal density function of } y}dy$$

7. $X, Y \rightarrow$ i.i.d cont random var f density function

$$P(X>Y) = P(X\in(t,\infty), Y\in(-\infty,t)) \quad \forall t\in\mathbb{R}$$

$$= \int_{-\infty}^{\infty}\left[\underbrace{\int_t^{\infty} f(u)\,du}_{\text{var in }t}\ \underbrace{\int_{-\infty}^{t} f(u)}_{\text{var in }t}\right]dt$$

$$= \int_{-\infty}^{\infty}\left[\int_t^{\infty} f(u)\,du - \left[\int_t^{\infty} f(u)\,du\right]^2\right]dt$$

as $X>Y$
$-\infty<y<\infty$
$-\infty<y<x$
$-\infty<x<\infty$

doubt

$$P(X>Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x)f(y)\,dy\,dx$$

$u = F(x)$
$du = f(x)dx$

$$= \int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{x} f(y)\,dy\right)dx$$

$$= \int_{-\infty}^{\infty} F(x)\,f(x)\,dx$$

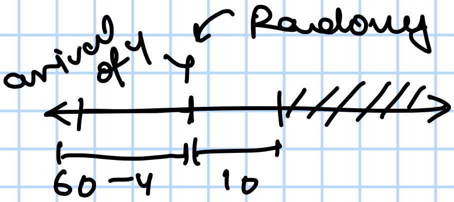
$$= \int_0^1 u\,du = \frac{1}{2}$$

8. $X \quad Y$

$f_X(x) = \frac{1}{60} \qquad f_Y(y) = \frac{1}{60}$

60

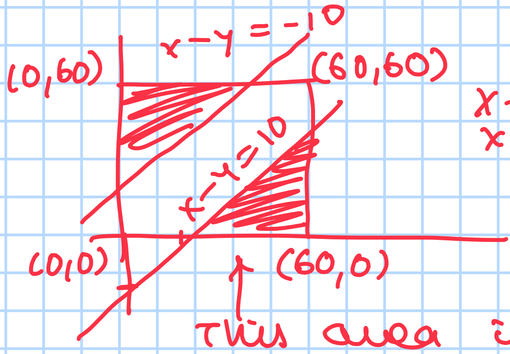
so $\int_0^{60}\frac{1}{60}dx = 1$



X: arrival time of man
Y: arrival time of women
doubt
X, Y iid uniform $[0, 60]$

$P(|X-Y|>10)$

where $f_{X,Y}(x,y) = \begin{cases} \frac{1}{3600}; & 0 \le x,y \le 60 \\ 0; & \text{otherwise} \end{cases}$



$x-y=10$
$x-y=-10$

This area is $\dfrac{\frac{1}{2}\times(50)^2\times 2}{3600} = \frac{25}{36}$

Total prob

or $P(|X-Y|>10)$

$= P(X-Y>10) + P(X-Y<-10)$

$= 2P(X-Y<-10)$

$= 2\int_{10}^{60}\int_{0}^{y-10} \frac{1}{3600}\,dx\,dy$

9. X, Y (ind random variables)

$X, Y \overset{iid}{\sim} \text{uniform}(1,2)$

To find : $f_{X+Y}(z) = P(X+Y \le z)$

Some value

Random variables

Step 1 : $\underbrace{f_{X,Y}(x,y)}_{\text{joint density } f^n} = f_X(x)\, f_Y(y)$

$= \begin{cases} 1; & \text{if } 1 \le x \le 2 \\ 0; & \text{o.w} \end{cases} \times \begin{cases} 1; & 1 \le y \le 2 \\ 0; & \text{otherwise} \end{cases}$

$= \begin{cases} 1; & 1 \le x \le 2 \text{ \& } 1 \le y \le 2 \\ 0; & \text{otherwise} \end{cases}$

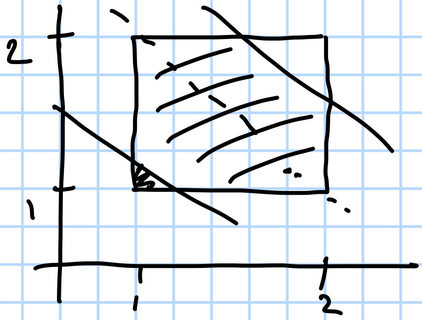
Step 2 : $Z = X+Y$

Z will be in $(2,4)$

$F_Z(z) = P(Z \le z)$

$= \begin{cases} 0; & z < 2 \\ \gamma(z); & 2 \le z \le 4 \\ 1; & z > 4 \end{cases}$

Step 3: graph:



$x + y = z$

$2 \le z \le 3$

$P(X+Y \le z) = \frac{\text{Area of triagle}}{\text{Area of sqr}}$

$= \frac{(z-2)^2}{2}$

$3 \le z \le 4$

$P(X+Y \le z) = 1 - \frac{\text{Area of t}}{\text{Area of Sqre}}$

$= 1 - \frac{(4-z)^2}{2}$

$$G_Z(z) = \begin{cases} 0 & ; \ z < 2 \\ \frac{(z-2)^2}{2} & ; \ 2 \le z < 3 \\ 1 - \frac{(4-z)^2}{2} & ; \ 3 \le z < 4 \\ 1 & ; \ z \geq 4 \end{cases}$$

$$f_Z(z) = \frac{d}{dz}(G_Z(z)) = \begin{cases} (z-2) ; & 2 \le z < 3 \\ (4-z) ; & 3 \le z < 4 \\ 0 ; & \text{ow} \end{cases}$$

10. $T_1, T_2, \ldots T_n$ ind &

$T_i \sim Exp(\lambda_i)$

To prove: $\min\{T_1 \ldots T_n\} \sim Exp(\lambda_1 + \cdots \lambda_n)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; \ x > 0, \lambda > 0 \\ 0 & ; \ \text{otheise} \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & ; \ x > 0 \\ 0 & ; \ \text{otheise} \end{cases}$$

$$= \begin{cases} \int_0^x \lambda e^{-\lambda t} dt ; & x > 0 \\ 0 & ; \ \text{otheise} \end{cases}$$

$Y = \min\{T_1, \ldots T_n\}$

$F_Y(y) = P(Y \le y)$

$= P(\min\{T_1 \ldots T_n\} \le y)$

$= 1 - P(\min\{T_1 \ldots T_n\} > y)$

$$= 1 - P(T_1 > y, \ldots T_n > y$$

$$= 1 - \prod_{i=1}^{n} P(T_i > y)$$

$$= 1 - \prod_{i=1}^{n} (1 - P(T_i \le y))$$

$$= 1 - \prod_{i=1}^{n} (1 - (1 - e^{-\lambda_i y}))$$

$$= 1 - \prod_{i=1}^{n} e^{-\lambda_i y}$$

$$= 1 - e^{-(\lambda_1 + \ldots \lambda_n) y}$$

$$\therefore \min\{T_1 \ldots T_n\} \sim Exp(\lambda_1 + \cdots \lambda_n)$$

Tutorial-7 :

1. $S \sim Exp(\lambda)$
$T \sim Exp(\mu)$
and S, T are independent

To show: $P(S<T) = \frac{\lambda}{\lambda+\mu}$

now, $S \sim Exp(\lambda)$
means that

$P(S>t) = e^{-\lambda t}$ for $t \geq 0$
$0 < \lambda < \infty$

$$f_S(t) = \begin{cases} \lambda e^{-\lambda t} & ; t \geq 0 \\ 0 & ; \text{otwise} \end{cases}$$

now $P(S \leq t) = \int_0^t f_S(t)\,dt$

$$= \int_0^t \lambda e^{-\lambda t}\,dt$$

$$= \lambda \left| \frac{e^{-\lambda t}}{-\lambda} \right|_0^t = \frac{\cancel{\lambda}}{\cancel{\lambda}}\left[1-e^{-\lambda t}\right]$$
$$= 1-e^{-\lambda t}$$

$P(S>t) = e^{-\lambda t}$; $t \geq 0$

now, $P(S<T)$ means

$$P(S<T) = \int_0^\infty \int_0^t \lambda \mu e^{-\lambda s} e^{-\mu t}\,ds\,dt$$

s value b/w 0 and t
for $T=t$

$$= \int_0^\infty \int_0^t \lambda \mu e^{-\lambda s} e^{-\mu t}\,ds\,dt$$

$$= \lambda\mu \int_0^\infty \left[\int_0^t [e^{-\lambda s}]\,ds\right] e^{-\mu t}\,dt$$

$$= \lambda\mu \int_0^\infty \left[\left| \frac{e^{-\lambda s}}{-\lambda} \right|_0^t \right] e^{-\mu t}\,dt$$

$$= \cancel{\lambda}\mu \int_0^\infty \left[1-e^{-\lambda t}\right] e^{-\mu t}\,dt$$

$$= \mu \int_0^\infty \left[e^{-\mu t} - e^{-(\lambda+\mu)t}\right] dt$$

$$= \mu\left[\frac{1}{\mu} - \frac{1}{\lambda+\mu}\right]$$

$$= 1 - \frac{\mu}{\lambda+\mu}$$

$$P(S<T) = \frac{\lambda}{\lambda+\mu}$$

2. $E_1, E_2 \ldots\ldots$ independent $Exp(\lambda)$

To prove :

$T_n = E_1 + E_2 + \cdots E_n$ has gamma (n, λ) distribution.

gamma(n,λ) is :

$X \sim$ gamma (α, λ)

density $\rightarrow$ $f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & ; x \geq 0 \\ 0 & ; \text{otherwise} \end{cases}$

$$\frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$$

$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy \quad \alpha > 0$

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(n) = n!$

$\Gamma(1) = 1$

now as $E_i \sim Exp(\lambda)$

$\sum_{i=1}^{n} E_i \sim$ gamma $(n, \lambda) \leftarrow$ To show

if $T \sim$ gamma (n, λ)
then

$M_T(t) = E[e^{Tt}]$

$$= \int_{-\infty}^{\infty} e^{xt} f(x)\, dx$$

$$= \int_0^\infty e^{xt} \frac{\lambda^n}{\Gamma n} x^{n-1} e^{-\lambda x} dx$$

$$\Gamma n = \int_0^\infty y^{n-1} e^{-y} dy \quad \underline{n > 0}$$

$$M_T(t) = \int_0^\infty \frac{\lambda^n}{\Gamma n} x^{n-1} e^{xt - \lambda x} dx$$

$$= \lambda^n \int_0^\infty \frac{1}{\Gamma n} x^{n-1} e^{\underbrace{+xt-\lambda x}} dx$$

$x(-\lambda+t) = -z$

$dz = -(-\lambda+t) = \lambda - t$

$dz = \lambda - t$

$$= \frac{-\lambda^n}{t-\lambda} \int_0^\infty \frac{1}{\Gamma n} \left[\frac{-z}{t-\lambda}\right]^{n-1} e^{-z} dz$$

$$= \frac{\lambda^n}{\lambda - t} \int_0^\infty + \frac{1}{\Gamma n} \left[\frac{z}{\lambda - t}\right]^{n-1} e^{-z} dz$$

$$= + \frac{\lambda^n}{(\lambda-t)^n} \cancel{\int_0^\infty \frac{1}{\Gamma n} z^{n-1} e^{-z} dz}$$

$$= + \left[\frac{\lambda}{\lambda - t}\right]^n$$

$$= (1 - t/\lambda)^{-n} \quad \text{for } |t| < \lambda$$

so $M_T(t) = (1 - t/\lambda)^{-n} \quad |t| < \lambda$

for gamma dist

now if $E \sim \text{gamma}(1, \lambda)$ then

$$f_E(x) = \begin{cases} \frac{\lambda^1}{\Gamma 1} x^{1-1} e^{-\lambda x} dx \,; & x \geq 0 \\ 0 & ; \text{otuise} \end{cases}$$

$$= \begin{cases} \lambda e^{-\lambda x} dx \,; & x \geq 0 \\ 0 & ; \text{otuise} \end{cases}$$

so $E_i \sim \text{gamma}(1, \lambda)$

now

$$M_{\Sigma E_i}(t) = \Pi M_{E_i}(t)$$

$$= \Pi (1 - t/\lambda)^{-1}$$

$$= (1 - t/\lambda)^{-n}$$

so $M_{\Sigma E_i}(t) = M_{T_n}(t)$

for $T_n \sim \text{gama}(n, \lambda)$

$\Rightarrow \Sigma E_i \sim \text{gamma}(n, \lambda)$

3. $X \sim N(0,1)$ $\quad X \sim N(\mu, \sigma^2)$

$\mu \to$ mean, $\sigma^2 \leftarrow$ variance

<u>To find</u>: $E(X^k)$

$$E(X^k) = \int_{-\infty}^{\infty} x^k f(x)\,dx$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$E(X^k) = \int_{-\infty}^{\infty} x^k \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-0)^2}{2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} x^k\,dx$$

$$= \begin{cases} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \underbrace{x^{2m+1}}_{\text{odd}} dx \,; & x = 2m+1 \\ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \underbrace{x^{2m}}_{\text{even}} dx \,; & x = 2m \end{cases}$$

$$= \begin{cases} 0 \,; & k = 2m+1 \\ 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} x^{2m}\,dx \,; & k = 2m \end{cases}$$

$$\frac{x^2}{2} = u \qquad x\,dx = du$$

$$\Gamma\alpha = \int_0^{\infty} x^{\alpha-1} e^{-x}\,dx$$

$$2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} x^{2m}\,dx$$

$$\cancel{2}\int_0^{\infty} \frac{1}{\sqrt{\cancel{2}\pi}} e^{-u} (2u)^m \frac{du}{(\cancel{2}u)^{1/2}}$$

$$= \frac{2^m}{\Gamma 1/2} \int_0^{\infty} e^{-u} (u)^{m-1/2}\,du$$

$$= \frac{2^m}{\Gamma 1/2} \int_0^{\infty} u^{\overset{\alpha}{(m+1/2)}-1} e^{-u}\,du$$

$$E(X^k) = \frac{2^m}{\Gamma 1/2} \Gamma(m+1/2)$$

for $k = 2m$

for $k = 2m+1$ is 0.

4. X is cauchy density f given by

$$f(x) = \frac{1}{\pi(1+x^2)} \quad ; \; -\infty < x < \infty$$

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\pi(1+x^2)} dx$$

$$= \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx$$

$$= 2 \int_{0}^{\infty} \frac{x}{\pi(1+x^2)} dx$$

$1 + x^2 = u$

$$= \int_{1}^{\infty} \frac{du}{\pi u} = \frac{1}{\pi} \ln|u| \Big|_{1}^{\infty}$$

as $\ln(u)$ for $u \to \infty$ diverges

$E(X)$ is not $< \infty$.

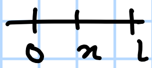
5. X is non-neg random variable

density of X is f
dist of X is F

$$F(x) = \int_{0}^{x} f(x)\, dx$$

$$\int_{0}^{\infty} 1 - F(x)\, dx = \int_{0}^{\infty} P(X \geq x) dx$$

$$= \int_{0}^{\infty} \left[\int_{x}^{\infty} f(t) dt \right] dx$$



$$= \int_{0}^{\infty} f(t) \left(\int_{0}^{t} 1 . dx \right) dt \quad 0 \leq x \leq t, \; x \leq t < \infty$$

$$= \int_{0}^{\infty} f(t).t.dt$$

$$\int_{0}^{\infty} 1 - F(x) dx = E(X)$$

$$= E(|X|) \quad \text{as } \underline{\underline{X \geq 0}}$$

$$E(|X|) < \infty \iff \int_{0}^{\infty} (1 - F(x)) dx < \infty$$

1. (X,Y) has

$$f(x,y) = \frac{1}{x} \qquad 0 \le y \le x \le 1$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^x \frac{1}{x}\,dy\,dx$$

$$= \int_0^1 \frac{x}{x}\,dx = 1$$

$$f(x,y) = \frac{1}{x} \qquad 0 \le y \le x \le 1$$

$$\int_0^x f(x,y)\,dy = f_X(x)$$

$$= \int_0^x \frac{1}{x}\,dy = 1$$

$$f(x) = \begin{cases} 1; & 0 \le x \le 1 \\ 0: & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{1/x}{1} = \frac{1}{x}$$

$$f_{Y|X}(y|x) = \begin{cases} 1/x & 0 \le y \le x \\ 0 & \text{o.w} \end{cases} \sim \text{unif}(0,x)$$

$$E(Y|X=x) = \psi(x) = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\,dy$$

$$= \int_0^x y\,\frac{1}{x}\,dy = \frac{x^2}{2x} = \frac{x}{2}$$

$$\psi(x) = \frac{x}{2}$$

$$E[Y|X] = \frac{X}{2}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

2. X & Y have JDF

$$f(x,y) = cx(y-x)e^{-y}$$

$$0 \le x \le y < \infty$$

(a) $$\int_0^{\infty}\int_0^{y} cx(y-x)e^{-y}\,dx\,dy$$

$$= \int_0^{\infty}\int_0^{y} ce^{-y}[yx - x^2]\,dx\,dy$$

$$= \int_0^\infty ce^{-y}\left[y.\frac{y^2}{2} - \frac{y^3}{3}\right]dy$$

$$= \int_0^\infty ce^{-y}\left[y^3/6\right]dy$$

$$= \frac{c}{6}\int_0^\infty y^3 e^{-y}dy$$

$$= \frac{c}{6}\sqrt{4} = 1$$

$$\Rightarrow c = \frac{6}{\sqrt{4}}$$

(b) $\int_0^y f(x,y)\,dx = f(y) = \frac{c}{6}y^3 e^{-y}$

$$f_{x/y}(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{\cancel{c}\,x(y-x)\cancel{e^{-y}}}{\cancel{c}/6\, y^3 \cancel{e^{-y}}}$$

$$= 6x(y-x)\,y^{-3}$$

$$0 \leq x < y$$

$$\int_x^\infty f(x,y)\,dy = f_X(x)$$

$$f_X(x) = \int_x^\infty cx(y-x)\,e^{-y}dy$$

$$= cx\int_x^\infty ye^{-y}dy - cx^2\int_x^\infty e^{-y}dy$$

$$= cx\left[y\int_x^\infty e^{-y}dy + \int_x^\infty e^{-y}\right] - cx^2\left[\frac{e^{-y}}{-1}\right]_x^\infty$$

$$= cx\left[xe^{-x} + e^{-x}\right] - cx^2\left[e^{-x}\right]$$

$$= \cancel{cx^2 e^{-x}} + cxe^{-x} - \cancel{cx^2 e^{-x}}$$

$$f_X(x) = cxe^{-x}$$

then $f_{Y|X}(y|x) = \dfrac{\cancel{cx}(y-x)\,e^{-y}}{\cancel{cx}\,e^{-x}}$

$$= (y-x)\,e^{-y+x}$$

$$0 \leq x \leq y < \infty$$

(c) $E(X|Y) = Y/2$

$E(Y|X) = X+2$

$$E(X|Y) = \int_0^y x\, 6x(y-x)\, y^{-3}\, dx$$

$$= \int_0^y 6x^2(y-x)y^{-3}dx$$

$$= 6y^{-3}\int_0^y yx^2 - x^3 dx$$

$$= 6y^{-3}\left[y\frac{y^3}{3} - \frac{y^4}{4}\right]$$

$$= 6y\left[\frac{1}{12}\right] = \frac{y}{2}$$

$$\Rightarrow E(X|Y) = Y/2$$

$$E(Y|X) = \int_x^\infty y(y-x)e^{x-y}\, dy$$

$$= e^x \int_x^\infty (y^2 - xy)\, e^{-y} dy$$

$$= e^x\left[\int_x^\infty y^2 e^{-y} - x\int_x^\infty y e^{-y} dy\right]$$

$$= \cancel{e^x}\left[x^2\cancel{e^{-x}} + 2x\cancel{e^{-x}} + 2\cancel{e^{-x}} - x\left[x\cancel{e^{-x}} + \cancel{e^{-x}}\right]\right]$$

$$= \cancel{x^2} + 2x + 2 - \cancel{x^2} - x$$

$$E(Y|X) = X+2$$

3. $f(x,y) = \lambda^2 e^{-\lambda y}$

$0 \le x \le y < \infty \quad , \lambda > 0$

$$f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y}\, dy$$

$$f_X(x) = \lambda e^{-\lambda x}$$

$$f_{Y|X}(y|x) = \lambda e^{-\lambda y + \lambda x}$$

$$E(Y|X) = \lambda\int_x^\infty y\, e^{-\lambda y + \lambda x}\, dy$$

$$= e^{\lambda x}\lambda \int_x^\infty y e^{-\lambda y} dy$$

$$= \cancel{\lambda} e^{\lambda x}\left[\frac{x e^{-\lambda x}}{\cancel{\lambda}} - \frac{e^{-\lambda x}}{\lambda\cancel{\lambda}}\right]$$

$$E(Y|X) = X - \frac{1}{\lambda}$$

4. $f(x,y)=\begin{cases} cx^2(8-y) & ; x \le y \le 2x, \quad 0 \le x \le 2 \\ 0 & ; \text{else} \end{cases}$

$$f_X(x) = \int_x^{2x} cx^2(8-y)\,dy$$

$0 \le x \le 2$

$$= cx^2\left[8x - \int_x^{2x} y\,dy\right]$$

$$= cx^2\left[8x - \frac{1}{2}(4x^2 - x^2)\right]$$

$$= cx^2\left[8x - \frac{3}{2}x^2\right]$$

$$f_X(x) = 8cx^3 - \frac{3}{2}cx^4$$

$$f_Y(y) = \int_0^2 cx^2(8-y)\,dx$$

$$= (8-y)(c)\int_0^2 x^2\,dx$$

$$= (8-y)(c)\left[\frac{1}{3}\right]\left[8\right]$$

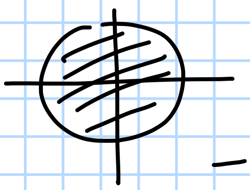
$$f_Y(y) = \frac{8}{3}(8-y)(c)$$

$$f_{X|Y}(x|y) = \frac{\cancel{c}x^2\cancel{(8-y)}}{\frac{8}{3}\cancel{(8-y)}\cancel{(c)}} = \frac{3}{8}x^2$$

$$f_{Y|X}(y|x) = \frac{\cancel{cx^2}(8-y)}{\cancel{cx^2}\left[8x - \frac{3}{2}x^2\right]}$$

$$f_{Y|X}(y|x) = \frac{8-y}{8x - \frac{3}{2}x^2}$$

5. $(X,Y) \rightarrow$ Random point



$x^2+y^2 < a^2$

$y^2 < a^2 - x^2$

$-\sqrt{a^2-x^2} < y < \sqrt{a^2-x^2}$

let's find

$$f(x,y) = \begin{cases} \frac{1}{\pi a^2} & ; x^2+y^2 \le a^2, \ |x| \le a, \ |y| \le a \\ 0 & ; \text{ow} \end{cases}$$

$$f_X(x) = \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{\pi a^2}\,dx = \frac{2}{\pi a^2}\left(\sqrt{a^2-x^2}\right), \quad |x| \le a$$

$$f_Y(y) = \frac{2\sqrt{a^2-y^2}}{\pi a^2}, \quad |y| \leq a$$

$$f_{Y|X}(y|x) = \frac{1}{2\sqrt{a^2-x^2}}$$

6. $$f(x,y) = \begin{cases} 6(1-x-y); & x \geq 0,\ y \geq 0,\ x+y \leq 1 \\ 0; & \text{else} \end{cases}$$

$$f_X(x) = \int_0^{1-x} 6(1-x-y)\,dy$$

$$= \left(6y - 6xy - 3y^2 \right|_0^{1-x}$$

$$= 6(1-x) - 6x(1-x) - 3(1-x)^2$$

$$= 6 - 6x - 6x + 6x^2 - 3(1+x^2-2x)$$

$$= 6 - 6x - \cancel{6x} + 6x^2 - 3 - 3x^2 + \cancel{6x}$$

$$= 3 - 6x + 3x^2$$

$$= 3(1-2x+x^2)$$

$$= 3(1-x)^2$$

$$\int_0^{1/2} f_{Y|X}(Y|X=1/2)\,y\,dy = \int_0^{1/2} \frac{\cancel{2}\cancel{6}(1-1/2-y)}{\cancel{3}(1/2)^2}\,dy$$

$$= 8\int_0^{1/2} (1/2-y)\,dy$$

$$= 8\left[\frac{1}{4} - \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right]$$

$$= 2 - 1 = 1$$

$$\text{cov}(X,Y) = E(X-\mu_X)(Y-\mu_Y)$$

$$= E(XY) - E(X)E(Y)$$

$$f_Y(y) = 3(1-y)^2$$

now

$$E(XY) = \int_0^1\int_0^{1-y} xy \cdot 6(1-x-y)\,dx\,dy$$

$$= 6\int_0^1 y\left(\int_0^{1-y} x - x^2 - xy\right)dx\,dy$$

$$= 6\int_0^1 y \int_0^{1-y} (x-x^2-xy)\,dx\,dy$$

$$= 6\int_0^1 y \quad \left[\left(x^2/2 - x^3/3 - x^2/2\, y \right) \right|_0^{1-y} dy$$

$$= 6\int_0^1 y \left[(1-y)\left(\frac{1-y}{2}\right)^2 - \frac{(1-y)^3}{3} \right] dy$$

$$= \cancel{6}\int_0^1 y \left(\frac{1}{\cancel{6}} \right) (1-y)^3\, dy$$

$$= \int_0^1 y\,(1-y)^3\, dy$$

$$= \int_0^1 ((1-y))\,(y)^3 dy$$

$$= \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

$$E(X) = \int_0^1 x \;\; 3(1-x)^2\, dx$$

$$= 3\int_0^1 (x)^2\,(1-x)\, dx$$

$$= 3\left[\frac{1}{3} - \frac{1}{4} \right] = \frac{1}{4}$$

$$E(Y) = \frac{1}{4}$$

$$E(XY) - E(X)E(Y) = \frac{1}{20} - \frac{1}{4} - \frac{1}{4}$$

$$= \frac{1}{20} - \frac{10}{20} = \frac{-9}{20}$$

7. $f(x,y) = \begin{cases} 8xy & ;\; 0 \le x \le y \;,\; 0 \le y \le 1 \\ 0 & ;\; \text{o.w} \end{cases}$

$$f(x) = \int_0^1 8yx\, dy = 8x\left(\frac{y^2}{2}\right)\Big|_0^1$$

$$= 4x$$

$$f(y) = \int_0^y 8xy\, dx = 8y\left(\frac{x^2}{2}\right)\Big|_0^y = 4y^3$$

<u>Not ind</u>.

8. (X,Y)

$$f(x,y) = \begin{cases} \frac{6-x-y}{8} & ;\; 0 < x < 2 \,,\; 2 < y < 4 \\ 0 & ;\; \text{o.w} \end{cases}$$

$$P(X \le x, Y \le y) = \int_0^y \int_0^x f(x,y)\, dx\, dy$$

$$f_X(x) = \int_2^4 \frac{6-x-y}{8}\, dy$$

$$= \left(\frac{6-x}{8}\right)(2) - \frac{1}{8}\left(\frac{1}{2}\right)(4^2-2^2)$$

$$= \frac{6-x}{4} - \frac{1}{8}\cdot\frac{1}{\cancel{2}}\cdot(\cancel{6})(\cancel{2})\,3$$

$$f_X(x) = \frac{3-x}{4}$$

$$f_Y(y) = \int_0^2 \frac{6-x-y}{8}\, d$$

$$= \left(\frac{6-y}{\cancel{8}4}\right)(\cancel{2}) - \frac{1}{\cancel{8}2}\frac{1}{2}(\cancel{4})$$

$$= \frac{6-y}{4} - \frac{1}{4} = \frac{5-y}{4}$$

$$f(x,y) = \frac{6-x-y}{8}$$

$$f_X(x) = \frac{3-x}{4}$$

$$f_Y(y) = \frac{5-y}{4}$$

$$f_{X|Y}(x|y) = \frac{\frac{6-x-y}{8}}{\frac{5-y}{4}}$$

$$f_{X|Y}(x|y) = \left(\frac{6-x-y}{5-y}\right)\left(\frac{1}{2}\right)$$

$$P(X<1 \mid Y=3) = \int_0^1 f_{X|Y}(x|3)\, dx$$

$$= \int_0^1 \left[\frac{6-x-3}{2}\right]\frac{1}{2}\cdot dx$$

$$= \int_0^1 \frac{3-x}{4}\, dx$$

$$= 3/4 - 1/4 \times 1/2 \times 1$$

$$= 3/4 - 1/8 = 5/8$$

$$P(X<1 \mid Y=3) = 5/8$$

$$P(X<1 \mid Y<3) = \int_2^3 \left(\int_0^1 \frac{6-x-y}{10-2y}\, dx\right) dy$$

$$= \int_2^3 \frac{1}{10-2y}\left[6 - \tfrac{1}{2} - y\right] dy$$

$$= \int_2^3 \frac{1}{10-2y}\left(\tfrac{11}{2} - y\right) dy$$

$$= \frac{1}{4}\int_2^3 \frac{1}{5-y}\left(11-2y\right) dy$$

$$= \frac{1}{4}\int_2^3 \frac{11-2y}{5-y}\, dy$$

$$5 - y = u$$
$$-dy = du$$

$$= \frac{1}{4}\int_2^3 \frac{11-2(5-u)}{u}\, du$$

$$= \frac{1}{4}\int_2^3 \frac{11-10+2u}{u}\, du$$

$$= \frac{1}{4}\int_2^3 \left(\frac{1}{u} + 2\right) du$$

$$= \frac{1}{4}\left[2 + \ln|u|\Big|_2^3\right]$$

$$= \frac{1}{4}\left[2 + \ln(3/2)\right]$$

$$= \tfrac{1}{2} + \tfrac{1}{4}\ln(3/2)$$

9. $X \sim$ uniform $(0,1)$

$f_{Y|X}(Y \mid X=x) \sim$ uniform $(0, x)$

$f_X(x) = 1 \quad 0 < x \leq 1$

$f_{Y|X}(Y \mid X=x) = \frac{1}{x}$

$0 < y \leq x$

$f(x,y) = f_X(x)\, f_{Y|X}(Y \mid X=x)$

$$f(x,y) = \begin{cases} 1/x \ ; & 0 \leq y \leq x \leq 1 \\ 0 \ ; & \text{o.w} \end{cases}$$

$$f_Y(y) = \int_y^1 1/x \, dx = \ln(1) - \ln(y) = \ln(1/y)$$

Tutorial-8 :

1. X+Y density in the case

(a) X is ind of Y
$X, Y \sim U(0,1)$

(b) X is ind of Y
$X, Y \sim Exp(\lambda)$

$$F_Z(z) = P(Z \leq z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\,dx$$

$x + y = z$
$-\infty \leq y \leq z-x$
$-\infty \leq x \leq \infty$

$y = v - x$
$dy = dv - dx$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{z} f(x, v-x)\,dv\,dx$$

$$F_Z(z) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} f(x, v-x)\,dx\,dv$$

$$f(z) = \int_{-\infty}^{\infty} f_{xy}(x, z-x)\,dx$$

diff

if ind then

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\,dx$$

$$f(x,y) = \begin{cases} 1; & 0<x,\ y<1 \\ 0; & \text{o.w} \end{cases}$$

$u = x+y$
$v = x-y$
$x = \frac{u+v}{2} = h_1(u,v)$
$y = \frac{u-v}{2} = h_2(u,v)$

$0 < x < 1$
$0 < y < 1$
$0 < \frac{u+v}{2} < 1 \Rightarrow 0 < u+v < 2$
$0 < \frac{u-v}{2} < 1 \Rightarrow 0 < u-v < 2$

$$J\left(\frac{x, y}{u, v}\right) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$$

$$f(u,v) = \begin{cases} 1\cdot(|-1/2|) ; & 0 < u+v < 2, \\ 0 ; & \text{o.w} \end{cases}$$

$$f(u) = \int_v 1/2\,dv = \int_{\max\{-u,\ u-2\}}^{\min\{u,\ 2-u\}} 1/2\,dv = \begin{cases} u & ; 0<u<1 \\ 2-u & ; 1 \leq u < 2 \\ 0 & ; \text{o.w} \end{cases}$$

$u = x+y$

$$f_{x+y}(u) = \begin{cases} u & ; \ 0<u<1 \\ 2-u & ; \ 1\leq u<2 \\ 0 & ; \ \text{O.w} \end{cases}$$

$X, Y \sim \text{Exp}(\lambda)$

$f(x) = \lambda e^{-\lambda x}$
$x>0$

$f(y) = \lambda e^{-\lambda y}$
$y>0$

$f(x,y) = \lambda^2 e^{-\lambda x - \lambda y}, \ x,y>0$

now $\quad x = \frac{u+v}{2} \qquad u+v>0$

$\quad y = \frac{u-v}{2} \qquad u-v>0$

$$f(u,v) = \left| J\left(\frac{x,y}{u,v}\right)\right| \lambda^2 e^{-\lambda u}$$

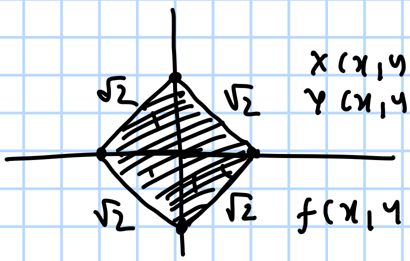
$$f(u,v) = \frac{1}{2}\lambda^2 e^{-\lambda u}; \quad u>v>-u$$

$$f(u) = \int_{-u}^{u} \frac{1}{2}\lambda^2 e^{-\lambda u}\, dv$$

$$f(u) = \frac{u\lambda^2 e^{-\lambda u}}{\Gamma(2)} \qquad u>0$$

$X+Y \sim \text{gamma}(2,\lambda)$

2.



$X(x,y) = x$
$Y(x,y) = y \qquad x,y \sim \text{uniform}(R)$

$f(x,y) = \frac{1}{\text{Area}} = \frac{1}{2}$

$f(x,y) = 1/2, \quad |x|+|y|<1$

$|y|<1-|x|$

$$f_x(x) = \int_{-(1-|x|)}^{(1-|x|)} \frac{1}{2}\, dy = \frac{1-|x|}{\cancel{2}}.\cancel{2} = 1-|x|$$

$f_Y(y) = 1-|y|$

Also not independent as

$f_X(x) f_Y(y) \neq f(x,y)$

method of transformation:

case I> x is a r.v with desity $f_x(x)$
$g: \mathbb{R} \to \mathbb{R}$ is one-one

Aim: To find $f_Y(y)$ where $Y = g(X)$

Subcase I> g is inc

$$F_Y(y) = P(g(X) \leq y)$$
$$= P(X \leq g^{-1}(y))$$
$$= F_X(h(y))$$

$$h(y) = g^{-1}(y)$$
$$f_Y(y) = f_X(h(y)) \frac{d}{dx}(h(y))$$

Subcase II> g is dec

$$F_Y(y) = P(g(X) \leq y)$$
$$= P(X \geq h(y))$$
$$= 1 - P(X \leq h(y))$$
$$f_Y(y) = -f_X(h(y)) \frac{d}{dy} h(y)$$
$$= f_X(h(y)) \underbrace{\left(-\frac{d}{dy} h(y)\right)}_{>0}$$

so $$f_Y(y) = f_X(h(y)) \left| \frac{d}{dy} h(y) \right|$$

$h(y) = g^{-1}(y)$
for $Y = g(X)$

case 2: $X_1, X_2 \ldots \ldots X_n$
joint dist $f_X(z)$

$g_i : \mathbb{R}^n \to \mathbb{R}$ is one-one
$\forall i = 1, 2 \ldots n$
$g_i(X_1, X_2, \ldots X_n) = Y_i$

now $\exists$ invertible fn's h_i's
s.t
$$X_1 = h_1(Y_1, \ldots Y_n)$$
$$X_2 = h_2(Y_1, \ldots Y_n)$$
$$\vdots$$
$$X_n = h_n(Y_1, \ldots Y_n)$$

insted of $\frac{dh(y)}{dy}$ we need determinant

$\left(\frac{\partial x_i}{\partial y_i}\right) \to$ Jacobian

$$J\left(\frac{x_1, \ldots x_n}{y_1, \ldots y_n}\right) = \det\left[\left(\frac{\partial x_i}{\partial y_j}\right)\right]$$

$$f_{Y_1, Y_2 \ldots Y_n}(y_1, \ldots y_n) = f_{X_1 \ldots X_n}(h_1(y), h_2(y), \ldots h_n(y))|J|$$

Example: if to find X_1+X_2

$$g_1(X_1, X_2) = Y_1$$
$$g_2(X_1, X_2) = Y_2$$

s.t both are one-one

$$X_1 + X_2 = Y_1$$
$$g_2(X_1, X_2) = Y_2 \rightarrow \text{auxiliary transformation}$$

s.t g_2 is one-one and

$$|J| \neq 0$$

then

$$\int_{Y_2} f_{Y_1, Y_2}(u, v)\, dv = f_{Y_1}(u)$$

as $Y_1 = X_1 + X_2$
$f_{Y_1}(u)$ is required

3. (a) $2X + 5Y$ density

$f_{X,Y}(x, y)$ = joint density of X and Y

$$g_1(X, Y) = 2X + 5Y = U$$
$$g_2(X, Y) = X = V$$

$$X = h_1(2X+5Y, X)$$
$$Y = h_2(2X+5Y, X)$$

now $X = V = h_1(U, V)$
$Y = \frac{U - 2V}{5} = h_2(U, V)$

now $\left| J\left(\frac{X, Y}{U, V}\right) \right| = \left\| \begin{matrix} 0 & 1 \\ 1/5 & -2/5 \end{matrix} \right\| = 1/5$

now $f_{U,V}(u, v) = \left(\frac{1}{5}\right) f_{X,Y}(X, Y)$

$$f_{U,V}(u,v) = \left(\frac{1}{5}\right) f_{X,Y}\left(v, \frac{u-2v}{5}\right)$$

$$f_U(u) = \int_{-\infty}^{\infty} \frac{1}{5} f_{X,Y}\left(v, \frac{u-2v}{5}\right) dv$$

(b) Joint density of $X+Y$ & $X-Y$

now, $g_1(X, Y) = X + Y = U$
$g_2(X, Y) = X - Y = V$

$$X = \frac{U+V}{2} = h_1(U,V)$$
$$Y = \frac{U-V}{2} = h_2(U,V)$$

now $J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \frac{1}{2}$

$$\Rightarrow f_{U,V}(u,v) = \frac{1}{2} f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

$f_U(u)$ is $X+Y$ density

$f_V(v)$ is $X-Y$ density

4. $(X_1, X_2, \ldots X_n)$ joint density $f(x_1, x_2, \ldots x_n)$

$$Y_j = \sum_{i=1}^{j} X_i$$

$f_{Y_1, \ldots Y_n}(y_1, y_2 \ldots y_n)$

$$g_1(X_1, X_2 \ldots X_n) = X_1 = Y_1$$
$$g_2(X_1, X_2 \ldots X_n) = X_1 + X_2 = Y_2$$
$$\vdots$$
$$g_n(X_1, X_2, \ldots X_n) = X_1 + X_2 + \cdots X_n = Y_n$$

$$X_1 = Y_1$$
$$X_2 = Y_2 - Y_1$$
$$X_3 = Y_3 - X_2 - X_1 = Y_3 - Y_2 + Y_1 - Y_1 = Y_3 - Y_2$$
$$X_4 = Y_4 - Y_3$$

$$X_i = Y_i - Y_{i-1}$$
$$X_1 = Y_1$$

now $J\left(\frac{X_1, X_2 \ldots X_n}{Y_1, Y_2 \ldots Y_n}\right) = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 \cdots & 0 \\ 0 & 0 & -1 & 1 \; 0 \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ & & & \cdots & 0 \\ & & & -1 & 1 \end{vmatrix} = 1$

so $f_{Y_1, Y_2 \ldots Y_n}(y_1, y_2 \ldots y_n)$

$$= f_{X_1, X_2 \ldots X_n}(y_1, y_2 - y_1, y_3 - y_2, \ldots y_n - y_{n-1})$$

5. $(X_1, X_2) \rightarrow$ cont. random vectors

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)} & ; \; x_1 > 0, \; x_2 > 0 \\ 0 & ; \text{o.w} \end{cases}$$

$$Y_1 = X_1 + X_2$$
$$Y_2 = \frac{X_1}{X_1 + X_2}$$

$$g_1(X_1, X_2) = Y_1 = X_1 + X_2$$
$$g_2(X_1, X_2) = Y_2 = \frac{X_1}{X_1 + X_2}$$

now $Y_1 = X_1 + X_2$
$Y_1 Y_2 = X_1$
$Y_1 = Y_1 Y_2 + X_2$
$X_2 = Y_1(1 - Y_2)$

now, $x_1 = y_1 y_2$
$x_2 = y_1(1 - y_2)$

$$J\left(\frac{x_1, x_2}{y_1, y_2}\right) = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = \left| -y_1 \cancel{y_2} - (y_1 - \cancel{y_1 y_2}) \right|$$
$$= y_1$$

$$f_{Y_1, Y_2}(y_1, y_2) = y_1 \, f_{X_1, X_2}(y_1 y_2, y_1(1 - y_2))$$

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} y_1 e^{-(y_1)} & ; \ 0 < y_1 < \infty, \ 0 < y_2 < 1 \\ 0 & ; \ \text{o.w} \end{cases}$$

$$f_{Y_1}(y) = \begin{cases} y_1 e^{-y_1} & ; \ 0 < y_1 < \infty \\ 0 & ; \ \text{o.w} \end{cases}$$

$$f_{Y_2}(y) = \begin{cases} 1 & ; \ 0 < y_2 < 1 \\ 0 & ; \ \text{o.w} \end{cases}$$

$Y_1 \sim \text{gamma}(\alpha = 2, \lambda = 1)$

$Y_2 \sim \text{uniform}(0, 1)$

6. (X, Y)

$$f(x, y) = \begin{cases} x e^{-(x+y)} & ; \ x > 0, \ y > 0 \\ 0 & ; \ \text{else} \end{cases}$$

(a) $Z = \frac{X}{Y}$ density

$$f_X(x) = \int_0^\infty x e^{-(x+y)} \, dy = x e^{-x}$$

$$f_Y(y) = \int_0^\infty x e^{-(x+y)} \, dx = e^{-y} [0 + 1] = e^{-y}$$

$$P(Z \leq z) = P\left(\frac{X}{Y} \leq z\right) = \int_0^\infty \int_0^{zy} f_{X,Y}(x, y)\,dx\,dy$$

$0 < y < \infty$

$0 < x < zy$

$$= \int_0^\infty \int_0^{zy} x e^{-(x+y)}\,dx\,dy$$

$$= \int_0^\infty e^{-y} \int_0^{zy} x e^{-x}\,dx\,dy$$

$$= \int_0^\infty e^{-y} \left[\frac{x e^{-x}}{-1} - e^{-x} \right]_0^{zy} dy$$

$$= \int_0^\infty e^{-y} \left(\left[\frac{zy e^{-zy} + e^{-zy}}{-1} \right] - [-1] \right) dy$$

$$= \int_0^\infty e^{-y} \left[1 - zy e^{-zy} - e^{-zy} \right] dy$$

$$= 1 - z \left[\int_0^\infty y e^{-(z+1)y}\,dy \right] - \left[\int_0^\infty e^{-(z+1)y}\,dy \right]$$

$$= 1 - z \left[\frac{1}{(z+1)^2} \right] - \frac{1}{(z+1)}$$

$$= 1 - \frac{z}{(z+1)^2} - \frac{(z+1)}{(z+1)^2}$$

$$= 1 - \frac{(2z+1)}{(z+1)^2}$$

$$= \frac{z^2 + \cancel{1} + \cancel{2z} - \cancel{2z} - \cancel{1}}{(z+1)^2}$$

$$= \frac{z^2}{(z+1)^2} = \left(\frac{z}{z+1} \right)^2$$

$$P(Z \leq z) = \left(\frac{z}{z+1} \right)^2$$

$$f_Z(z) = \frac{d}{dz}\left(\left(\frac{z+1-1}{z+1} \right)^2 \right)$$

$$= 2\left(\frac{z}{z+1} \right) \left[\frac{1}{(z+1)^2} \right]$$

$$f_Z(z) = \frac{2z}{(z+1)^3}$$

(b) Z by introducing $W = X + Y$

now, $Z = \frac{X}{Y}$ $\quad W = X + Y$

then

$X = ZY$

$W = ZY + Y$

$\Rightarrow Y = \frac{W}{Z+1}$

$\Rightarrow X = \frac{ZW}{Z+1}$

now

$$J\left(\frac{x, y}{z, w}\right) = \begin{vmatrix} z/z+1 & w\left[1/(z+1)^2\right] \\ 1/z+1 & w\left[-1/(z+1)^2\right] \end{vmatrix}$$

$$= \left|-wz/(z+1)^3 - w(z+1)^3\right|$$

$$= \frac{w(z+1)}{(z+1)^3} = \frac{w}{(z+1)^2}$$

$$f_{Z,W}(z, \omega) = \frac{w}{(z+1)^2} f_{X,Y}\left(\frac{z\omega}{z+1}, \frac{\omega}{z+1}\right)$$

$$= \frac{w}{(z+1)^2}\left(\frac{z\omega}{z+1} e^{-(\omega)}\right)$$

$$= \frac{z\omega^2}{(z+1)^3} e^{-\omega}$$

$x > 0$

$y > 0$

$\frac{\omega}{z+1} > 0$ $\qquad \frac{z\omega}{z+1} > 0$

$\omega > 0$ $\qquad z > 0$

$$f_Z(z) = \int_0^\infty \frac{z\omega^2}{(z+1)^3} e^{-\omega}$$

$$= \frac{z}{(z+1)^3} \Gamma(3) = \frac{2z}{(z+1)^3}$$

7. X, Y are iid $N(0,1)$

density of X/Y

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-(x)^2/2}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$Z = X/Y$

$W = Y$ then

$X = ZW$
$Y = W$

$$J\left(\frac{X, Y}{Z, W}\right) = \begin{vmatrix} Z & W \\ 1 & 0 \end{vmatrix} = W$$

$$f_{Z,W}(z, \omega) = |w| \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

$$= |W| \frac{1}{2\pi} e^{-(z^2\omega^2 + \omega^2)/2}$$

$$= |w| \frac{1}{2\pi} e^{-\frac{(z^2+1)}{2}\omega^2}$$

$$f_Z(z) = \int_0^{\infty} 2\omega \frac{1}{\cancel{2}\pi} e^{-\frac{(z^2+1)}{2}\omega^2} d\omega$$

$$= \frac{1}{\pi} \int_0^{\infty} \omega e^{-\frac{(z^2+1)}{2}\omega^2} d\omega$$

$\frac{(z^2+1)}{2}\omega^2 = v$

$\omega(z^2+1)\,d\omega = dv$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{1}{(z^2+1)} e^{-v} dv = \frac{1}{\pi(1+z^2)}$$

8. X is ind Y
$f(x)$ $g(y)$ then

$Z = XY$
$W = 1/Y$

then $Z = XY$
$W = 1/Y$

$X = ZW$
$Y = \frac{1}{W}$

$$J\left(\frac{X, Y}{Z, W}\right) = \begin{vmatrix} W & Z \\ 0 & -\frac{1}{W^2} \end{vmatrix} = \left|\frac{1}{W}\right|$$

$$f_Z(z) = \int_W f(z\omega)\, g\left(\frac{1}{\omega}\right) d\omega$$

gamma distribution

$X \sim gamma(\alpha, \lambda)$

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} ; & x > 0 \\ 0 ; & o.w \end{cases}$$

$\alpha > 0, \lambda > 0$

$\Gamma(\alpha) = (\alpha - 1)!$

$\Gamma(1/2) = \sqrt{\pi}$

Note: $\alpha = 1$

$$gama(\alpha = 1, \lambda) \quad f_X(x) = \begin{cases} \frac{\lambda^1}{\Gamma(\alpha)} x^{1-1} e^{-\lambda x} ; & x > 0 \\ 0 & ; ow \end{cases}$$

$$= \begin{cases} \lambda e^{-\lambda x} ; & x > 0 \\ 0 & ; ow \end{cases}$$

$gama(\alpha = 1, \lambda) \sim exp(\lambda)$

Note: $X_1, X_2, \dots X_n \sim Exp(\lambda)$
iid then
$X_1 + X_2 + \dots X_n \sim gamma(\alpha = n, \lambda)$

Note: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad \alpha > 0$

now $Y \sim gama(\alpha, \lambda)$

$$E(Y) = \int_0^\infty y \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y} dy$$

$$= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} y^\alpha e^{-\lambda y} dy$$

$$= \int_0^\infty \frac{(\lambda y)^\alpha e^{-\lambda y}}{\Gamma(\alpha)} dy$$

$\lambda y = u$
$\lambda dy = du$

$$= \frac{1}{\lambda} \int_0^\infty \frac{u^\alpha e^{-u} du}{\Gamma(\alpha)}$$

$$= \frac{1}{\lambda} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{(\alpha)!}{(\alpha-1)!} \frac{1}{\lambda} = \frac{\alpha}{\lambda}$$

$$E(Y) = \frac{\alpha}{\lambda}$$

$$var(Y) = E(Y^2) - [E(Y)]^2$$

$$= \int_0^\infty y^2 \frac{\lambda^\alpha y^{\alpha-1} e^{-\lambda y}}{\Gamma(\alpha)} dy - \frac{\alpha^2}{\lambda^2}$$

$$= \int_0^\infty \frac{\lambda^{\alpha+1}}{\lambda} \frac{y^{\alpha+1}}{\Gamma(\alpha)} e^{-\lambda y} dy - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{1}{\lambda} \int_0^\infty \frac{(\lambda y)^{\alpha+1} e^{-\lambda y}}{\Gamma(\alpha)} dy - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{1}{\lambda^2} (\alpha+1)(\alpha) - \frac{\alpha^2}{\lambda^2}$$

$$var(Y) = \frac{\alpha}{\lambda^2}$$

so if $\alpha = 1$ ($exp(\lambda)$) then

$$E(X) = \frac{1}{\lambda}$$

$$var(X) = \frac{1}{\lambda^2}$$

<u>Beta, distribution</u> : (first kind Beta)

$X \sim Beta_1(\alpha, \beta)$

$$f_X(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} ; & 0<x<1 \\ 0 & : ow \end{cases}$$

$$\beta(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}$$

$$X \sim Beta_1(1,1) \Rightarrow f_X(x) = \begin{cases} \frac{1}{B(1,1)} x^0(1-x)^0 ; & 0<x<1 \\ 0 & ; ow \end{cases}$$

$$f_X(x) = \begin{cases} 1 ; & 0<x<1 \\ 0 ; & o.w \end{cases}$$

or $X \sim unif(0,1)$

now $X \sim Beta_1(\alpha, \beta)$

$$E(X) = \int_0^1 \frac{(x)(x)^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} dx$$

$$= \frac{\beta(\alpha+1, \beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+1)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+1)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}}$$

$$E(X) = \frac{(\alpha)}{(\alpha+\beta)}$$

now $\operatorname{var}(X) = E(X^2) - (E(X))^2$

$$= \frac{\Gamma(\alpha+2)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+2)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}} - (E(x))^2$$

$$= \frac{(\alpha+1)(\alpha)}{(\alpha+\beta+1)(\alpha+\beta)} - \left[\frac{\alpha}{\alpha+\beta}\right]^2$$

$$= \left(\frac{\alpha+1}{\alpha+\beta+1} - \frac{\alpha}{\alpha+\beta}\right)\left(\frac{\alpha}{\alpha+\beta}\right)$$

$$= \left(\frac{\cancel{\alpha^2} + \cancel{\alpha\beta} + \cancel{\alpha} + \beta - \cancel{\alpha^2} - \cancel{\alpha\beta} - \cancel{\alpha}}{(\alpha+\beta+1)(\alpha+\beta)}\right)\left(\frac{\alpha}{\alpha+\beta}\right)$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Note: $E(X) = \frac{\alpha}{\alpha+\beta}$

$\operatorname{var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Beta distribution of second kind: $(Beta_2(\alpha, \beta))$

$$f_X(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1+x)^{-\alpha-\beta} ; & 0 < x < \infty \end{cases}$$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$X \sim Beta_1(\alpha,\beta)$ then $\frac{X}{1-X} \sim Beta_2(\alpha,\beta)$

$X \sim Beta_2(\alpha,\beta)$, $\frac{X}{1+X} \sim Beta_1(\alpha,\beta)$

Cauchy: $X \sim C(\mu, \sigma)$

σ → Scale parameter

$\mu \in \mathbb{R}$, $\sigma > 0$

μ → location parameter

$$f_X(x) = \frac{1}{\pi}\frac{\sigma}{(x-\mu)^2+\sigma^2} \quad \forall x \in \mathbb{R}$$

Standard Cauchy: $X \sim C(0,1)$ or

$$f_X(x) = \frac{1}{x^2+1}$$

Note: $F(X \le x) = \int_{-\infty}^{x} \frac{1}{\pi} \frac{\sigma}{(x-\mu)^2+\sigma^2} dx$

$$= \int_{-\infty}^{x} \frac{1}{\pi} \frac{1}{\sigma} \times \frac{1}{\left[\left(\frac{x-\mu}{\sigma}\right)^2+1\right]} dx$$

$\frac{x-\mu}{\sigma} = u$

$\frac{dx}{\sigma} = du$

$$= \int_{-\infty}^{\sigma u+\mu} \frac{1}{\pi} \frac{1}{(u^2+1)} du = \frac{1}{\pi}\left[\tan^{-1}[u]_{-\infty}^{\sigma u+u}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1}(u)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{x-\mu}{\sigma}\right)\right]$$

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-\mu}{\sigma}\right)$$

Note: X_1, X_2 iid $N(0,1)$

then $\frac{X_1}{X_2} \sim C(0,1)$

Note: No moment if $r > 1$

$$E(|X|) = \frac{1}{\pi}\int_{-\infty}^{\infty} |x| \frac{1}{1+x^2} dx = \frac{1}{\pi}\int_0^{\infty} \frac{2x}{1+x^2} dx = \frac{1}{\pi}\int_0^{\infty} \frac{1}{t} dt$$

$E|X|^r$

for $r<1$

is done as we get $Beta_2(\alpha, \beta)$

Gaussian or Normal:

$X \sim N(\mu, \sigma^2)$

$\sigma > 0$

$\mu \in \mathbb{R}$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \forall x \in \mathbb{R}$$

Standard normal dist is $N(0,1)$

k^{th} order: $E(X^k) = \begin{cases} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2m} e^{-x^2/2} dx & ; k = 2m \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2m+1} e^{-x^2/2} dx & ; k = 2m+1 \end{cases}$

$\rightarrow 0$ as odd function

$$E(X^{2m}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2m} e^{-x^2/2}\, dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \int_{0}^{\infty} x^{2m} e^{-x^2/2}\, dx$$

$$= \frac{(2)^{1/2}}{\Gamma(1/2)} \int_{0}^{\infty} x^{2m} e^{-x^2/2}\, dx$$

$$\frac{x^2}{2} = y$$

$$x\,dx = dy$$

$$= \frac{\cancel{(2)^{1/2}}}{\Gamma(1/2)} \int_{0}^{\infty} (2y)^m e^{-y} \frac{dy}{(\cancel{2}y)^{1/2}}$$

$$= \frac{2^m}{\Gamma(1/2)} \int_{0}^{\infty} (y)^{m-1/2} e^{-y}\, dy$$

$$= \frac{2^m}{\Gamma(1/2)} \Gamma(m+1/2)$$

<u>Note</u> : <u>Box-Muller transformation</u> :

$$X, Y \sim \text{Uniform}(0,1)$$

$$U = \sqrt{-2\ln X} \cos(2\pi Y)$$

$$V = \sqrt{-2\ln X} \sin(2\pi Y)$$

are iid $N(0,1)$

let $U = \sqrt{-2\ln X} \cos(2\pi Y)$

$V = \sqrt{-2\ln X} \sin(2\pi Y)$

$$U^2 + V^2 = -2\ln X$$

$$X = e^{-(U^2+V^2)/2}$$

$$\frac{V}{U} = \tan(2\pi Y)$$

$$2\pi Y = \tan^{-1}\left(\frac{V}{U}\right)$$

$$Y = \frac{1}{2\pi} \tan^{-1}\left(\frac{V}{U}\right)$$

$$X = e^{-\frac{(U^2+V^2)}{2}}$$

$$f_{X,Y}(x,y) = 1 \quad 0 \le x \le 1,\ 0 \le y \le 1$$

$$X = e^{-\frac{(u^2+v^2)}{2}}$$

$$Y = \frac{1}{2\pi} \tan^{-1}\left(\frac{V}{U}\right)$$

now,

$$\frac{dX}{du} = e^{-\frac{(u^2+v^2)}{2}} (-2u)$$

$$\frac{dX}{dv} = e^{-\frac{(u^2+v^2)}{2}} (-2v)$$

$$\frac{dy}{du} = \frac{1}{2\pi}\left(\frac{1}{1+\left(\frac{v}{u}\right)^2}\right)\left(\frac{-1}{u^2}\right)v$$

$$\frac{dy}{dv} = \frac{1}{2\pi}\left(\frac{1}{1+\left(\frac{v}{u}\right)^2}\right)\left(\frac{1}{u}\right)$$

moment generating function of a random variable :

$$M_X(t) = E(e^{tX}) = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx & ; X \text{ is cont random variable} \\ \sum_{x=0}^{\infty} e^{tx} P(X=x) & ; X \text{ is non-neg disc random var} \end{cases}$$

$\exists\, h>0$, s.t $\forall\, t\in(-h, h)$

$E(e^{tX})$ exist in this open interval

Note : if expectation does not exist, then moment generating function does not exist

$$M_X(t) = E(e^{tX}) = E\left[\sum_{r=0}^{\infty}\frac{(tX)^r}{r!}\right]$$

$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r)$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots$$

so to find r-th moment of X
we find $\frac{t^r}{r!}$ coeff in $M_X(t)$ extension

Note : $\left.\frac{d}{dt}M_X(t)\right|_{t=0} = E(X)$

$$\left.\frac{d^2}{dt^2}M_X(t)\right|_{t=0} = E(X^2)$$

$$\left.\frac{d^r}{dt^r}M_X(t)\right|_{t=0} = E(X^r)$$

Note : if CDF, MGF pair

$(F_X, M_X(t))$ & $(F_Y, M_Y(t))$
then
if $M_X(t) = M_Y(t)$
$\Rightarrow F_X(z) = F_Y(z)$

Note: X_i's are ind

$$M_{X_1+X_2+\dots X_n}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

Note: MGF of X is $M_X(t)$ then

$aX+b$ will have

$e^{b} M_X(at)$ as MGF for $a \neq 0$

Characteristic function:

as MGF not always exist
fourier transformation/characteristic functions do exist

$$\phi_x(t) = E(e^{itX})$$

$$= \begin{cases} \int_{-\infty}^{\infty} e^{itx} f_X(x)\,dx \ ; & X \text{ is c.r.v} \\ \sum_{x=0}^{\infty} e^{itx} P(X=x) \ ; & X \text{ is non-neg d.r.v} \end{cases}$$

Name	Support	$f_X(x)$	$E(X)$	$Var(X)$	$\phi_X(t)$
$Ber(p)$	$\{0,1\}$	$p^x(1-p)^{1-x}$	p	$p(1-p)$	$[1-p+pe^{it}]$
$Bin(n,p)$	$\{0,1,\dots n\}$	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$	$[1-p+pe^{it}]^n$
Geometric(p)	$\{1,2,\dots\}$	$p(1-p)^{x-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-(1-p)e^{it}}$
$Poi(\lambda)$	$\{0,1,\dots\}$	$\frac{\lambda^x e^{-\lambda}}{\lambda!}$	λ	λ	$e^{\lambda(e^{it}-1)}$
$N(\mu,\sigma^2)$	$\mathbb{R}$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$	μ	σ^2	$e^{it\mu+i^2(t^2\sigma^2)/2}$
$Exp(\lambda)$	$(0,\infty)$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$[1-(it/\lambda)]^{-1}$
Gamma(α,λ)	$(0,\infty)$	$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$[1-(it/\lambda)]^{-\alpha}$
Beta$_1$ (α,β)	$(0,1)$	$\frac{1}{B(\alpha,\beta)}(x)^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$E[X^k] = \frac{\Gamma(\alpha+k)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+k)\Gamma(\alpha)}$
Beta$_2$ (α,β)	$(0,\infty)$	$\frac{1}{B(\alpha,\beta)}(x)^{\alpha-1}(1+x)^{-\alpha-\beta}$	$\frac{\alpha}{\beta-1}$	$\frac{\alpha(\alpha+\beta-1)}{(b-1)^2(b-1)}$	$E[X^k] = \frac{\Gamma(\alpha+k)\Gamma(b-k)}{\Gamma(\alpha)\Gamma(b)}$
Cauchy (μ,σ)	$\mathbb{R}$	$\frac{1}{\pi}\frac{\sigma}{(x-\mu)^2+\sigma^2}$	—	—	$e^{it\mu-\sigma\lvert t\rvert}$